Coverings (for arcwise connected, locally arcconnected)

Def Let X be a topological space. A covering space of X is a pair consisting of a space $\tilde{X}$ and a cont map $p: \tilde{X} \longrightarrow X$ such that the following condition holds: Each point $x \in X$ has an arc-connected open neighbourhood U s.t. each arc component of $p^{-1}(U)$ is mapped topologically onto U by p. It is assume that $p^{-1}(U)$ non-empty. Such a neighborhood is called an elementary neighborhood

Examples i) $p: \mathbb{R} \longrightarrow S^1$, $t \longmapsto (\cos t, \sin t)$ continuous, neigborhoods like $U_{\pi/4} = \{(\cos(x), \sin(x)) \mid t - \pi/4 \le x \le t + \pi/4\}$ are elementary neighborhoods

ii) $p: S^1 \longrightarrow S^1$ and $n > 0$ integer (it works equally for $n < 0$) $(\cos(t), \sin(t)) \longmapsto (\cos(nt), \sin(nt))$. The same neighborhoods are elementary neighborhoods

iii) $p: X \xrightarrow{\text{homeomorphism}} Y$ $\quad p^{-1}(U) = V$ (arc-connected as well)

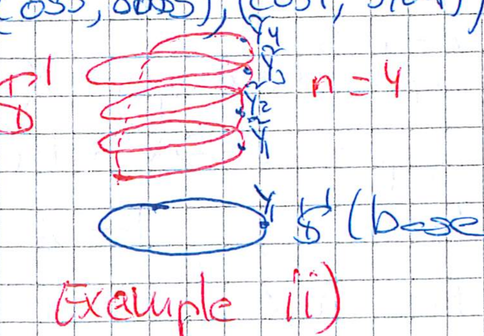
iii) If $p_1: \tilde{X} \longrightarrow X$ and $p_2: \tilde{Y} \longrightarrow Y$ are covering maps, then $p_1 \times p_2: \tilde{X} \times \tilde{Y} \longrightarrow X \times Y$ $\quad (p_1 \times p_2)((\tilde{x}, \tilde{y})) = (p_1 \tilde{x}, p_2 \tilde{y})$ is a covering map. How are the elementary neighborhoods ?!

iv) Ex: $p: \mathbb{C} \longrightarrow S^1 \times S^1 = \mathbb{T}$ covering map p

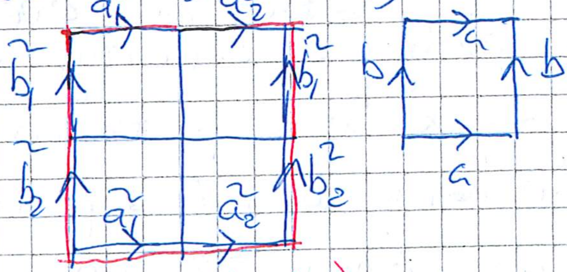
$(s+it) \longmapsto ((\cos s, \sin s), (\cos t, \sin t))$

$p_1: \mathbb{T} \longrightarrow \mathbb{T}$ covering map

$((\cos s, \sin s), (\cos t, \sin t)) \longmapsto ((\cos(2s), \sin(2s)), (\cos(2t), \sin(2t)))$



Example ii)



Example iv)

v) Co

Using Example iii) we have this two examples for complex functions

v) $p_n(z): \mathbb{C} \longrightarrow \mathbb{C}$ given by $p_n(z) = z^n$
$(\mathbb{C} \setminus \{0\}, p_n)$ covering of $\mathbb{C} \setminus \{0\} = \{r \in \mathbb{R}, r > 0\} \times S^1$
using polar coordinates $\mathbb{C} \setminus \{0\}$ wraps n times around itself

vi) Consider $\exp(z): \mathbb{C} \longrightarrow \mathbb{C}$ $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
$(\mathbb{C}, \exp)$ covering of $\mathbb{C} \setminus \{0\} = \{r \in \mathbb{R}, r > 0\} \times S^1$ ($\mathbb{R} \times \mathbb{R}$)
considering $\exp(z) = e^x(\cos y + i \sin y)$ for $z = x + iy$
We can see the covering map as a product of the homeomorphism $p_1: \mathbb{R} \longrightarrow \mathbb{R}^+$ $p_1(x) = e^x$ and the covering $p_2: \mathbb{R} \longrightarrow S^1$ $p_2(y) = (\cos y, \sin y)$.
We can see directly that $U_z = \{w \in \mathbb{C} ; |w - z| < |z|\}$ is an elementary neighborhood

Def A cont map: $f: X \longrightarrow Y$ is a local homeomorphism if each pt $x \in X$ has an open neighborhood V s.t $f(V)$ open and f maps topologically V onto $f(V)$
Then covering maps are local homeomorphisms (because the spaces are locally arc-connected)
The converse is not true in general: consider the local homeomorphism $f: (0, 9) \longrightarrow S^1$ $f(t) = (\cos t, \sin t)$
What happen at $(1, 0) \in S^1$??

Paths in Covering Spaces

Let $(\tilde{X}, p)$ be a covering space of a base space X
Let $g: I \longrightarrow \tilde{X}$ be a path in $\tilde{X}$, then $pg: I \longrightarrow X$ path in X. We have seen (as p cont.) that if $g_0, g_1: I \longrightarrow X$ s.t $g_0 \simeq g_1$ then $pg_0 \simeq pg_1$ for paths in X

Lemma Let $(\tilde{X}, p)$ be a covering space of X, $\tilde{x}_0 \in \tilde{X}$ and $x_0 = p(\tilde{x}_0)$. Then, for any path $f: I \longrightarrow X$ with $f(0) = x_0$, there is a unique path $g: I \longrightarrow \tilde{X}$ $g(0) = \tilde{x}_0$ and $pg \simeq f$.

also

Lemma Let $(\tilde{X}, p)$ be a covering space of X and let $g_0, g_1 : I \longrightarrow \tilde{X}$ be paths in $\tilde{X}$ which have the same initial point. If $pg_0 \simeq pg_1$ then $g_0 \simeq g_1$; in particular g_0 and g_1 have the same terminal pt.

As a result of the previous lemmae, we have

Lemma If $(\tilde{X}, p)$ is a covering space of X, then the sets $p^{-1}(x)$ for all $x \in X$ have the same cardinality.

As a consequence:

Th Let $(\tilde{X}, p)$ be a covering space of X, $\tilde{x}_0 \in \tilde{X}$ and $x_0 = p(\tilde{x}_0)$. Then $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ monomorphism ($p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ subgroup of $\pi_1(X, x_0)$)

Now, consider $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ with $p(\tilde{x}_0) = p(\tilde{x}_1) = x_0$

$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$

$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) \leq \pi_1(X, x_0)$ by considering a path $g: I \longrightarrow \tilde{X}$ $g(0) = \tilde{x}_0$, $g(1) = \tilde{x}_1$ with class γ we obtain an isomorphism $u: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(\tilde{X}, \tilde{x}_1)$

$\alpha \longmapsto \gamma^{-1}\alpha\gamma$

$$\begin{array}{ccc} \pi_1(\tilde{X}, \tilde{x}_0) & \xrightarrow{p_*} & \pi_1(X, x_0) \\ u \downarrow & & \downarrow v \\ \pi_1(\tilde{X}, \tilde{x}_1) & \xrightarrow{p_*} & \pi_1(X, x_0) \end{array}$$

; $v(\beta) = (p_*\gamma)^{-1}\beta(p_*\gamma)$ where $p_*\gamma \in \pi_1(X, x_0)$

So $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = (p_*\gamma)^{-1}(p_*(\pi_1(\tilde{X}, \tilde{x}_0)))(p_*\gamma)$

conjugate subgroups of $\pi_1(X, x_0)$

In the case of $\mathbb{R} \longrightarrow S^1$ and $\mathbb{C} \longrightarrow T$ we have that the trivial group is a subgroup of $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$

Now consider a topological space Y (connected and locally arc-connected) and a continuous map $f: Y \longrightarrow X$ (with $f(y_0) = x_0$). Let $(\tilde{X}, p)$ be a covering space of X, with $x_0 = p(\tilde{x}_0)$. When can we assure that the map $f: Y \longrightarrow X$ <u>lifts</u> to a cont map $\tilde{f}: Y \longrightarrow \tilde{X}$ ($\tilde{f}(y_0) = \tilde{x}_0$)?

<u>Th</u> Let $Y, X, (\tilde{X}, p) \ldots$ be in the above conditions. Then, given a map $f: (Y, y_0) \xrightarrow{\text{cont}} (X, x_0)$, there exists a lifting $\tilde{f}: (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ ($\tilde{f}(y_0) = \tilde{x}_0$) iff

$f_* \pi_1(Y, y_0) \leq p_* \pi_1(\tilde{X}, \tilde{x}_0)$

$$\begin{tikzcd} e: \pi_1(Y, y_0) \arrow[r, "\tilde{f}_*"] \arrow[dr, "f_*"'] & \pi_1(\tilde{X}, \tilde{x}_0) \arrow[d, "p_*"] \\ & \pi_1(X, x_0) \end{tikzcd}$$

Observe that $\tilde{f}$ is unique

<u>Example</u>: Topological groups. Assume that G is a topological space, $e \in G$ given and two continuous maps $\mu: G \times G \longrightarrow G$ and $c: G \xrightarrow{\text{homeo}} G$ such that for any $x \in G$ $\mu(x, e) = \mu(e, x) = x$
$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z)$
μ associativ and $\mu(x, c(x)) = \mu(c(x), x) = e$ for any $x \in G$. Then $\pi_1(G, e)$ is an abelian group.

Example $(S^1, \cdot)$, $(S^3, \cdot)$ are topological groups
$S^1 = \{e^{it} : t \in \mathbb{R}\}$, $\mu(e^{it_1}, e^{it_2}) = e^{i(t_1 + t_2)}$ (rotations of $\mathbb{C}$)
usual identification
($S^3 = \{q \in \mathbb{H}am : |q| = 1\}$ $q_1 q_2$ is the quaternion-multiplication.

Now if G is a arcwise connected and locally arc-connected and $p: \tilde{G} \longrightarrow G$ is a covering map and $\tilde{e}$ s.t. $p(\tilde{e}) = e \in G$ Then there is a unique $\underset{\text{cont}}{\text{lift}}$ (multiplication) $\tilde{\mu}: \tilde{G} \times \tilde{G} \longrightarrow \tilde{G}$ commuting the diagram and $\tilde{G}$ is a topological group p is continuous and a homomorphism.

$$\begin{tikzcd} \tilde{G} \times \tilde{G} \arrow[r, "\tilde{\mu}"] \arrow[d, "p"'] \arrow[d, shift left=3, "p"] & \tilde{G} \arrow[d, "p"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd}$$

Def Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering spaces of X. A homomorphism $\varphi: \tilde{X}_1 \to \tilde{X}_2$ is a cont. map making the diagram commutative

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & \circlearrowleft & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd} \qquad p_1 = p_2 \varphi$$

A homomorphism φ is an isomorphism if there exists $\psi: \tilde{X}_2 \to \tilde{X}_1$ s.t $p_2 = p_1 \psi$ and $\psi\varphi = Id_{\tilde{X}_1}$, $\varphi\psi = Id_{\tilde{X}_2}$

$\varphi: \tilde{X}_1 \xrightarrow{iso} \tilde{X}_1$ is an automorphism of the covering (a homeomorphism of $\tilde{X}$, commuting with p). Automorphisms form a group: the deck-transformation group $A(\tilde{X}, p)$

Now, the group $A(\tilde{X}, p)$ acts without fixed points on $\tilde{X}$ (if $\varphi \neq Id_{\tilde{X}}$, φ has no fixed pts) Because if $\varphi(x) = x = Id_{\tilde{X}}(x)$ for some pt in $\tilde{X}$, we will have two liftings of Id_X

$$\begin{tikzcd} \tilde{X} \arrow[r, "{\varphi, Id_{\tilde{X}}}"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{X} \arrow[d, "p"] \\ X \arrow[r, "Id_X"] & X \end{tikzcd}$$

with a common element then $\varphi = Id_{\tilde{X}}$

Now, consider when we can lift continuous maps, we have:

Let $(\tilde{X}_1, p_1)$, $(\tilde{X}_2, p_2)$ coverings of X and $\tilde{x}_1 \in \tilde{X}_1$, $\tilde{x}_2 \in \tilde{X}_2$ s.t $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$. Then, there exist a homomorphism $\varphi: (\tilde{X}_1, p_1) \longrightarrow (\tilde{X}_2, p_2)$ s.t $\varphi(\tilde{x}_1) = \tilde{x}_2$ iff

$$p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) \leq p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$$

φ is an isomorphism iff $p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$

So we have:

Th Two covering $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ of X are isomorphic iff for any two points $\tilde{x}_1 \in \tilde{X}_1$, $\tilde{x}_2 \in \tilde{X}_2$ s.t $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x_0$, the subgroups $p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1)$ and $p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$ are conjugate in $\pi_1(X, x_0)$

Now consider $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ coverings of X. If $\varphi : (\tilde{X}_1, p_1) \xrightarrow{\text{homo}} (\tilde{X}_2, p_2)$ then $\varphi : (\tilde{X}_1, \varphi) \longrightarrow \tilde{X}_2$ is a covering

Consider $x \in X$ and U elementary neighborhood of x for p_1 and p_2. We have to prove i) φ surjective : take $y \in \tilde{X}_2$ choose base pts $x_1 \in \tilde{X}_1$, $x_2 = \varphi(x_1)$; $x_0 = p_1(x_1) = p_2(x_2)$

Choose a path f in $\tilde{X}_2$ starting at x_2 and finishing at y; let $g = p_2 f$ the "quotient" path in X and h the lifting (unique) to $\tilde{X}_1$

$$\tilde{X}_1 \xrightarrow{\varphi} \tilde{X}_2$$

$$x_1 \xrightarrow{h} h(1) \in p_1^{-1}(g(1)) \qquad x_2 \xrightarrow{f} y = f(1)$$

$$\tilde{X}_1 \xrightarrow{p_1} X$$

$$x_0 \xrightarrow{g} g(1)$$

let $\tilde{x} = h(1)$

$p_2(y) = p_1(\tilde{x})$

and $\varphi(\tilde{x}) = y$

Now, an elementary neighborhood for $y \in \tilde{X}_2$ is given by the connected component of $p_2^{-1}(U)$ containing y; where $x = p_2(y)$ and U the elementary neighborhood as above

The universal covering

If there exist a covering $(\tilde{X}, p)$ of X, such that $\pi_1(\tilde{X}, \tilde{x}) = \{1\}$, then given a covering $(\tilde{X}', p')$ of X

Clearly $p_* \pi_1(\tilde{X}, \tilde{x}) = \{1\} \leq p'_* \pi_1(\tilde{X}', \tilde{x}')$ $(p(\tilde{x}) = p'(\tilde{x}') = x)$ and we have a homomorphism $\varphi : (\tilde{X}, p) \longrightarrow (\tilde{X}', p')$ so $\varphi : (\tilde{X}, p) \longrightarrow \tilde{X}'$ a covering. As the trivial group is unique in $\pi_1(X)$. A covering $(\tilde{X}, p)$ with $\pi_1(\tilde{X}) = \{1\}$ is called a universal covering and two universal coverings are isomorphic (and homeomorphic as top. spaces).

Example i) $\mathbb{R}$ is the universal covering of S^1

ii) $\mathbb{C}$ the universal covering of T

Important fact: Let X be a space with universal covering $(\tilde{X}, p)$. If (X_1, p_1) is a covering space of X and (X_2, p_2) is a covering space of X_1, then $(X_2, p_1 p_2)$ is a covering of X

$$\begin{tikzcd} \tilde{X} \arrow[rr, "p"] \arrow[dr, "p_{X_1}"] \arrow[d, "p_{X_2}"'] & & X \\ X_2 \arrow[r, "p_2"'] & X_1 \arrow[ur, "p_1"'] & \end{tikzcd}$$

Think of the elementary neighborhoods of any $x \in X$

This fact has implications in finding holomorphic functions between Riemann surfaces

Now we can see that $A(\tilde{X}_1, p_1)$ of a covering of a space X, we have to look at the action of $\pi_1(X, x)$ on the fibre $p^{-1}(x)$: the monodromy. So

Def Let $(\tilde{X}, p)$ be a covering of X and $x \in X$. For any point $\tilde{x} \in p^{-1}(x)$ and any $\alpha \in \pi_1(X, x)$ we define $\alpha(\tilde{x}) = \tilde{x} \cdot \alpha \in p^{-1}(x)$ as follows. By the results of lifting paths in X, given the homotopy class α of a loop f based at x, there is a unique class $\tilde{\alpha}$ of paths (possibly not loops) in $\tilde{X}$ with representative the unique lifting of f, $\tilde{f}: I \longrightarrow \tilde{X}$ with $\tilde{f}(0) = \tilde{x}$, as $p\tilde{f}(1) = x$ so $\tilde{f}(1) \in p^{-1}(x)$ and we can define $\tilde{x} \cdot \alpha = \tilde{f}(1)$

(observe that any representative is homotopic and gives the same image of $\tilde{x}$)

It is easy to control (do it!) that

i) $\tilde{x} \cdot 1 = \tilde{x}$, notice that $1 \in \pi_1(X, x)$ is the class of the constant loop x.

ii) $(\tilde{x} \cdot \alpha) \cdot \beta = \tilde{x} \cdot (\alpha\beta)$. S

So $\pi_1(X, x)$ acts (on the right) on $p^{-1}(x)$. $\pi_1(X, x)$ acts transitively, i.e. Given $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$ there is

$\alpha \in \pi_1(X, x)$ s.t. $\tilde{x}_1 \cdot \alpha = \tilde{x}_2$. This is true because $\tilde{X}$ is arcwise connected so given $\tilde{x}_1$ and $\tilde{x}_2$ there is a path class $\tilde{\alpha}$ in $\tilde{X}$ starting at $\tilde{x}_1$ and finishing at $\tilde{x}_2$. Let $\alpha \in p_*(\tilde{\alpha})$, $\tilde{x}_1 \cdot \alpha = \tilde{x}_2$. As $p(\tilde{x}_1) = p(\tilde{x}_2) = x$ $\alpha \in \pi_1(X, x)$.

Of course $\pi_1(X,x)$ can be consider a group of permutations of the points in $p^{-1}(x)$

As an example the universal covering $p: \mathbb{R} \to S^1$ $t \mapsto (\cos t, \sin t)$
$1 = (1, 0) \in S^1$; $\pi_1(S^1, 1) = \mathbb{Z}$ act as the bijections of $\mathbb{Z} = \{ \dots, -n, \dots, -1, 0, 1, \dots, n, \dots \}$

Given the action of a group G (in our case $\pi_1(X,x)$) on a set Ω and a point $\omega \in \Omega$ the isotropy group of ω is $H_\omega = \{ g \in G : \omega \cdot g = \omega \}$ (in fact $H \leq G$)
Control it!!

For coverings $p: \tilde{X} \to X$ and $\tilde{x} \in p^{-1}(x)$

$$H_{\tilde{x}} = p_*(\pi_1(\tilde{X}, \tilde{x}))$$ since any class $\tilde{\alpha}$ of ~~s.t. $\tilde{x} \cdot \alpha$~~ paths in $\tilde{X}$ must be the class of a loop based at $\tilde{x}$.

We have

Th For any automorphism $\varphi : (\tilde{X}, p) \to (\tilde{X}, p)$ of the covering, any point $\tilde{x} \in p^{-1}(x)$ and $\alpha \in \pi_1(X, x)$

$$\varphi(\tilde{x} \cdot \alpha) = (\varphi \tilde{x}) \cdot \alpha$$

To see it, Consider a representative $f: I \to X$ of α and lift ($f(0) = f(1) = x$) it to the covering $\tilde{f} : I \to \tilde{X}$
$\tilde{f}(0) = \tilde{x}$ and $\tilde{f}(1) \in p^{-1}(x)$. Consider the class of $\tilde{f}$, $\tilde{\alpha}$
$p_*(\tilde{\alpha}) = \alpha$ and $\tilde{x} \cdot \alpha = \tilde{f}(1)$. Now consider $\varphi(\tilde{x})$ and a lifting $\tilde{g}$ of f with $\tilde{g}(0) = \varphi(\tilde{x})$ and $\tilde{g}(1) = \varphi(\tilde{x} \cdot \alpha)$. But

$$p_*[\varphi_*(\tilde{\alpha})] = (p\varphi)_*[\tilde{\alpha}] \overset{\varphi \text{ automorphism}}{=} p_*(\tilde{\alpha}) = \alpha$$

Consider $\varphi_*(\alpha)$ is the lifting $\tilde{g}$ of the path f also and $\varphi_*(\tilde{x})\cdot\alpha$ is the terminal pt. of $\varphi_*(\alpha)$, so

$$\varphi(\tilde{x}\cdot\alpha) = \tilde{g}(1) = \varphi(\tilde{x})\tilde{\cdot}\alpha$$

As a consequence. Let $p:\tilde{X}\to X$ be a covering then $A(\tilde{X},p)$ is naturally isomorphic to the group of automorphisms of $p^{-1}(x)$ as a $\pi_1(X,x)$-set

Notice that $p:\tilde{X}\to X$, $x\in X$ and $p^{-1}(x)$

We have $H_{\tilde{x}} \cong p_*\pi_1(\tilde{X},x) \leq \pi_1(X,x)$

So we can identify $p^{-1}(x)$ with a transversal for $p_*\pi_1(\tilde{X},x)$ in $\pi_1(X,x)$, a transversal is the space of cosets of a subgroup in a group.

As a consequence given a covering $p:\tilde{X}\to X$ for any point $x\in X$ and $\tilde{x}\in p^{-1}(x)$, $A(\tilde{X},p)$ is isomorphic to the quotient group

$$N(p_*\pi_1(\tilde{X},\tilde{x}))/p_*\pi_1(\tilde{X},x)$$

where $N(p_*\pi_1(\tilde{X},x))$ is the normalizer of $p_*(\pi_1(\tilde{X},\tilde{x}))$ in $\pi_1(X,x)$.

We say that a covering $p:\tilde{X}\to X$ is regular if the subgroup $p_*\pi_1(\tilde{X},\tilde{x})$ is normal in $\pi_1(X,x)$ ($x\in X$, $\tilde{x}\in p^{-1}(x)$). Such a covering is called regular

Example i) As $\pi_1(T,x)=\mathbb{Z}\times\mathbb{Z}$ is abelian, any subgroup of it is normal in $\mathbb{Z}\times\mathbb{Z}$ and any covering of the torus is regular.

ii) If $p:\tilde{X}\to X$ is the universal covering of X with $\pi_1(\tilde{X})=\{1\}$ then $p_*\{1\}=\{1\}$ normal subgroup of $\pi_1(X,x)$ (for any point $x\in X$)

Then the universal covering is a regular covering

For the case of normal subgroups H of a group G, the set of coset is a group itself: the quotient gr G/H

$(1 \to H \to G \twoheadrightarrow H/G \twoheadrightarrow 1)$

So, if $p: \tilde{X} \to X$ is a regular covering of X, $p_*(\pi_1(\tilde{X}, \tilde{x}))$ normal in $\pi_1(X, x)$ $(\tilde{x} \in p^{-1}(x))$

and $A(\tilde{X}, p) = \pi_1(X, x) / p_*(\pi_1(\tilde{X}, \tilde{x}))$

for any $x \in X$ and any $\tilde{x} \in p^{-1}(x)$

If $p: \tilde{X} \to X$ is the universal covering $A(\tilde{X}, p) = \pi_1(X, x)$

Example: Consider again the universal covering

$p: \mathbb{R} \to S^1$, $A(\mathbb{R}, p) = \mathbb{Z}$ as explained before

ii) The universal covering $p: \mathbb{C} \to \mathbb{T}$ has

$A(\mathbb{C}, p) = \underbrace{\mathbb{Z} \times \mathbb{Z}}_{\text{a lattice}} = \{n z_1 + m z_2;\ z_1, z_2$ lin. independent complex numbers, $n, m \in \mathbb{Z}\}$

We have seen that $\mathbb{T} = \mathbb{C} / A(\mathbb{C}, p)$

As for the monodromies: As $\pi_1(X, x)$ acts on $p^{-1}(x)$ transitively we have an epimorphism

$$\Theta : \pi_1(X, x) \longrightarrow \Sigma_{p^{-1}(x)}$$

$\Sigma_{p^{-1}(x)}$ symmetric gr. on $p^{-1}(x)$

where $\Theta(p_*(\pi_1(\tilde{X}, \tilde{x}))) = H_{\tilde{x}}$, when we see $\tilde{x}$ as a symbol, remember $H_{\tilde{x}} = \{\alpha \in \pi_1(X, x)\ \alpha(\tilde{x}) = \tilde{x}\}$

$H_{\tilde{x}} = Stb_{\pi_1(\tilde{X}, \tilde{x})}(\tilde{x})$ (the transitivity means that we can take any preimage)

So we can see $\pi_1(\tilde{X}, \tilde{x}) \cong p_*(\pi_1(\tilde{X}, x)) = \Theta^{-1}(H_{\tilde{x}})$

<u>Example</u>: Consider the 4-sheeted covering of $\mathbb{T}$ given by $p: \tilde{X} \xrightarrow{4:1} \mathbb{T}$ with monodromy

$\Theta : \pi_1(\mathbb{T}, x) = \mathbb{Z} \times \mathbb{Z} = \langle a, b / aba^{-1}b^{-1} = 1 \rangle \longrightarrow \Sigma_4$

$\Theta(a) = (1\,2)(3,4)$, $\Theta(b) = (1,3)(2,4)$, so

$Stb_{\Sigma_4}(1) = \{Id\}$ and $\pi_1(\tilde{X}, x) = \ker\Theta = \mathbb{Z} \times \mathbb{Z}$

(in fact, we use $\mathbb{Z}/2 \cong 2\mathbb{Z}$) in fact $\Theta(\pi_1(\tilde{X}, x)) = C_2 \times C_2$

$2\mathbb{Z}$

On the other hand, given a covering $p: \tilde{X} \to X$. The automorphism group $A(\tilde{X}, p)$ acts transitively on $p^{-1}(x)$ iff p is a regular covering

In general as $p: \tilde{X} \to X$ open map; we can see X as a quotient space of $\tilde{X}$, but $X = \tilde{X}/A(\tilde{X},p)$ iff $p: \tilde{X} \to X$ is a regular covering; equivalently $A(\tilde{X}, p)$ acts transitively on $p^{-1}(x)$, for any $x \in X$

As for the converse if Y is a topological space, (connected, locally arc-connected) and G is a group of homeomorphisms of Y that acts without fixed pts and the orbit of any $x \in X$ is discrete then the projection $p: Y \to Y/G$ is a regular covering with $A(Y, p) = G$. (Recall the orbit of $x := \{g(x) ; g \in G\}$)

Examples i) We have seen $\mathbb{Z} = \{n\tau ; n \in \mathbb{Z}\}$ τ translation of $\mathbb{R}$
and $\mathbb{Z} \times \mathbb{Z} = \{n\tau_1 + m\tau_2 ; n, m \in \mathbb{Z}\}$ discrete, τ_1, τ_2 lin. ind translations
translations are homeomorphisms as they are isometries of $\mathbb{R}$ resp. $\mathbb{R}^2 = \mathbb{C}$.

Finally a little word on existence of coverings
If a space X has universal covering $p: \tilde{X} \to X$ then given any conjugacy class of subgroups of $\pi_1(X, x)$; there exist a covering space $p_Y: Y \to X$ such that $p_{Y*}(\pi_1(Y, y))$ belongs to the conjugacy class ($p_Y(y) = x$)

So, we have to see in which conditions the

universal covering does exists. The condition is that every point $x \in X$ has a neighborhood U such that the homeomorphism $i_*: \pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial; $i_*(\pi_1(U, x)) = \{1\}$.
As manifolds, and manifolds with boundary, all satisfy this property, but they are (connected) and locally arc-connected. Manifolds admit universal covering. In fact we can think of the universal covering as the space of all possible liftings of paths in X.