Chapter FU (as in Fuchsian Groups)

As As you will show in the last exercise $PSL(2,\mathbb{R})$ is homeomorphic to $\mathbb{R}^2 \times S^1$
$PSL(2,\mathbb{R})$ is a topological group.

<u>Def</u> A Fuchsian group is a discrete subgroup of $PSL(2,\mathbb{R})$

Example: $PSL(2,\mathbb{Z})$ is a Fuchsian group

<u>Def</u> Let G be a group of homeomorphisms of a topological space X. Then G acts properly discontinuously on X iff each compact subset K of X, $g(K) \cap K = \emptyset$ except for a finite number of g in G.
In particular the stabilizer of each element G_x is finite

Consider $E = \{T \in PSL(2,\mathbb{R}) \mid T(w) \in K\}$ for a given $w \in \mathbb{H}$ and K a compact subset of $\mathbb{H}$. E is compact and if Γ is a Fuchsian group, being discrete, $\{T \in \Gamma \mid T(w) \in K\}$ is finite. Then we have

<u>Theorem</u> Let Γ be a subgroup of $PSL(2,\mathbb{R})$

i) Γ is Fuchsian gr iff Γ acts properly discontinuously on $\mathbb{H}$

ii) Let Γ be a Fuchsian gr. and let $p \in \mathbb{H}$ be fixed by some element $\gamma \in \Gamma$. Then there is a neighborhd W of p s.t <u>no</u> other point in W is fixed by a non-identity element of Γ

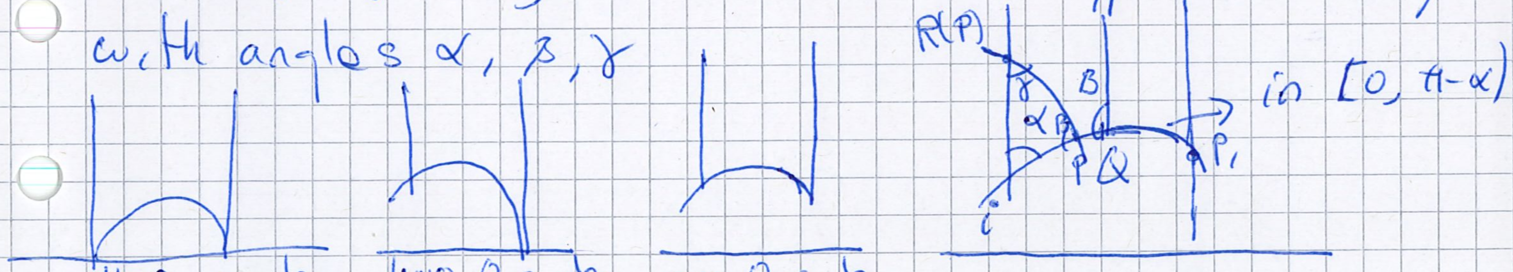
Proof of part ii). Let p be fixed by $\gamma \neq \mathrm{Id}$.
First of all as Γ discrete, given p take the disc $\overline{B_\varepsilon(p)}$ of p, compact, then $\{T \in \Gamma \mid T(p) \in \overline{B_\varepsilon(p)}\}$ finite and there exists $0 < \delta < \varepsilon$ s.t $\overline{B_\delta(p)}$ only contains p in the Γ-orbit of p. ~~take~~ take $V = B_{\delta/2}(p)$, if $V \cap \gamma(V) \neq \emptyset$ for

some $\gamma \in \Gamma$, then $\exists z \in V$ s.t $\gamma(z) \in V$, and
$d_H(z, p) < \delta/2$ and $d_H(\gamma(z), p) < \delta/2$ and
$d_H(p, \gamma(p)) < \delta$ (triangle inequality). ~~So $\gamma(p) \neq p$~~
Now, let p be fixed by an element T of Γ, there is neighborhd W of p s.t $W \cap T(W) \neq \phi$, then $T(p) = p$.
Take $q \in W$ fixed by some element $\gamma \neq Id$. $\gamma(W) \cap W \neq \phi$ and $\gamma(p) = p$, also $\gamma(q) = q$, then $p = q$.

As a consequence ~~if~~ Γ is a Fuchsian gr. iff $\forall z \in \mathbb{H}$, the Γ-orbit of z is a discrete subset of $\mathbb{H}$.

Observe that whether a discrete group acts properly discontinuously or not depends also on the space on which the group acts. For instance the modular gr. $PSL(2, \mathbb{Z})$ does not act properly discontinuously on $\mathbb{R} \cup \{\infty\}$ as the orbit of 0 is $\mathbb{Q} \cup \{\infty\}$, dense in $\mathbb{R} \cup \{\infty\}$

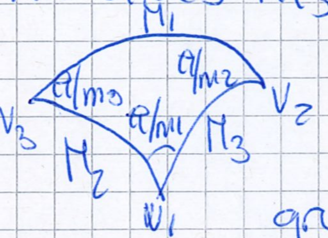
We know that given α, β, γ non-negative real numbers s.t $\alpha + \beta + \gamma < \pi$, then there exists a hyperbolic triangle with angles α, β, γ



all 0-angles | two 0-angles | one 0-angle

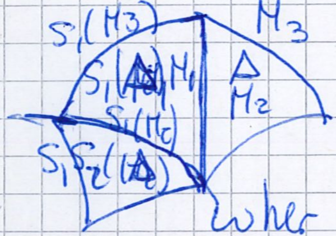
<u>Def</u> Let ℓ be an H-line. Then an H-reflection in ℓ is an H-isometry of $\mathbb{H}$, other than the identity, which fixes every pt in ℓ.
If ℓ_0 is the imaginary axis, then $S_0 : z \mapsto -\bar{z}$ is an H-reflection in ℓ_0. If ℓ is another H-line $\exists \gamma \in PSL(2, \mathbb{R})$ $\gamma(\ell) = \ell_0$. Then $S = T^{-1} S_0 T$ is a H-reflection in ℓ.

Given m_1, m_2, m_3 positive integers such that
$\pi/m_1 + \pi/m_2 + \pi/m_3 < \pi$ $\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1\right)$
there is a hyperbolic triangle Δ with angles π/m_1, π/m_2, π/m_3 (and sides M_1, M_2, M_3)



We have H-reflections S_i on H-lines containing M_i, $1 \leq i \leq 3$. Let $\overline{\Gamma}$ be the group generated by S_1, S_2, S_3 (not Fuchsian)

$\Gamma = \overline{\Gamma} \cap PSL(2,\mathbb{R})$ and $\overline{\Gamma} = \Gamma \cup \Gamma S_1$. The image of Δ under the H-reflection S_1 has sides $S_1(M_1) = M_1$, $S_1(M_2)$, $S_1(M_3)$ and $S_1 S_2 S_1^{-1} = S$ fixes $S_1(M_2)$, $S(S_1(\Delta)) = S_1 S_2(\Delta)$



The hyperbolic triangles surrounding V_3 are, Δ, $S_1(\Delta)$, $S_1S_2(\Delta)$, ..., $(S_1 S_2)^{m_3 - 1} S_1$ where $S_1 S_2$ is a rotation of angle $2\pi/m_3$ about V_3
One can show that $\{T(\Delta) \mid T \in \overline{\Gamma}\}$ forms a tessellation of $\mathbb{H}$ (Caratheodory, Magnus, Maskit, Beardon). The Γ-orbits are all discrete and Γ is a Fuchsian group:
<u>a triangle group</u> $\Gamma = \langle S_1 S_2 = X, S_2 S_3 = Y \mid X^{m_3} = Y^{m_1} = (XY)^{m_2} = 1 \rangle$

If Γ is a Fuchsian group, all of whose non-identity elements have the same fixed-point set, then Γ is cyclic. Then every abelian Fuchsian group is cyclic and, in particular, there is no Fuchsian group isomorphic to $\mathbb{Z} \times \mathbb{Z}$, so no torus is uniformised by $\mathbb{H}$. Neither by $\hat{\mathbb{C}}$ as no subgroup of $\mathrm{Aut}(\hat{\mathbb{C}})$ act without fixed-pts. So the universal covering of a RS of genus 1 is $\mathbb{C}$.

<u>Th</u> Given a non-cyclic Fuchsian gr. Γ, its normalizer $N_{PSL(2,\mathbb{R})}(\Gamma)$ is Fuchsian

Proof ~~Suppose~~ Let that $\Lambda = N_{PSL(2,\mathbb{R})}(\Gamma)$ not Fuchsian. Then Λ contains an infinite sequence $\{T_i\}$ of distinct elements s.t $T_i \to I$ as $i \to \infty$. Given $Id \neq S \in \Gamma$, $\{T_i S T_i^{-1}\} \xrightarrow[i\to\infty]{} S$. As Γ Fuchsian (discrete) $\exists m$, and $\forall i \geq m$

$T_i S T_i^{-1} = S$ and $T_i S = S T_i$, ~~and~~ T_i $i \geq m$ and S have the same fixed points. This for all $S' \in \Gamma$, then Γ is abelian and cyclic, a contradiction.

Def Let Γ be an arbitrary Fuchsian group and let $p \in \mathbb{H}$ be not fixed by any element of $\Gamma \setminus \{Id\}$.
The Dirichlet region of Γ centred at p is

$$D_p(\Gamma) = \{ z \in \mathbb{H} \mid d_H(z,p) \leq d_H(z, T(p)),\ \forall T \in \Gamma \}$$

$T(z) = \frac{az+b}{cz+d}$, $ad-bc=1$ $\quad D_p(\Gamma) \{ z \in \mathbb{H} \mid \frac{|T(z)-p|}{|z-p|} \geq \frac{1}{|cz+d|},\ T \in \Gamma \}$

* $D_p(\Gamma)$ is a connected fundamental region for Γ (p not fixed by any non-identity element of Γ)

Example $\Gamma = PSL(2,\mathbb{Z})$. $p = ki$, $k > 1$, p is not fixed by any non-identity element of Γ. Using $T(z) = z+1$, $T_1(z) = z-1$ we get $c=0$, $d=1$ and $|z \pm 1 - ki| \geq |z - ki|$ then $D_{ki}(\Gamma) \subseteq \{ z \in \mathbb{H} \mid -\frac{1}{2} \leq \operatorname{Re} z \leq \frac{1}{2} \}$. Using $T(z) = -\frac{1}{z}$

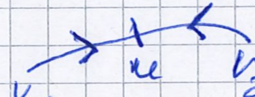
$$\frac{|-1/z - ki|}{|z - ki|} \geq \frac{1}{|z|} \quad \text{or} \quad |1 + kiz|^2 \geq |z - ki|^2, \text{ then } |z| \geq 1$$

So $D_{ki}(\Gamma) \subseteq \{ z \in \mathbb{H} \mid |z| \geq 1, |\operatorname{Re} z| \leq \frac{1}{2} \} = F$.
$D_{ki}(\Gamma)$ is symmetric with respect to the imaginary axis.
Now, if $z \in F$, $S \in \Gamma \setminus \{Id\}$ and $w = S(z) \in F$. Then, $z, w \in \partial F$ and either $z = w$ or z and w symmetric with respect to the imaginary axis

So $D_{ki}(\Gamma) = F = \{ z \in \mathbb{H} \mid |z| \geq 1, |\operatorname{Re} z| \leq \frac{1}{2} \}$

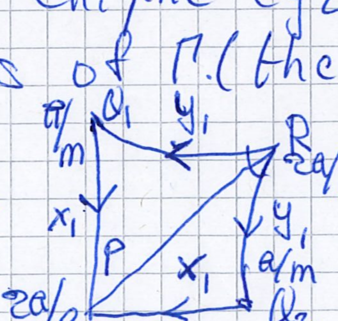
– Elliptic elements and Vertices of the Fund. Region.

Let F be the Dirichlet region of a Fuchsian gr Γ. Two vertices u, v in F are congruent if $\exists\, T \in \Gamma$ with $T(u) = v$ ($T^{-1}(v) = u$). Now u is fixed by an (elliptic) element T_1 of Γ iff v is fixed by an element ($T T_1 T^{-1} = T_2$) of Γ. Congruent vertices form a cycle. Vertices in a cycle are all fixed by elliptic elements or none. A cycle of vertices formed by vertices fixed by elliptic elements is called an "<u>elliptic cycle</u>". Clearly, every pt fixed $v \in \mathbb{H}$ fixed by an elliptic element $S \in \Gamma$ belongs to the boundary of $T(F)$ for some $T \in \Gamma$ and $u = T^{-1}(v) \in \partial F$ fixed by $\hat{S} = T^{-1} S T$; S and $\hat{S}$ of order k (finite). If $k \geq 3$ then u is a vertex of F of angle at most $2\pi/k$ (Observe that ∂F consists of points and H-segments $[v_1, v_2]$). If $k = 2$ then u belongs to the interior of a side, wher $\hat{S}$ the segments $[v_1, u]$ and $[v_2, u]$



We can consider that $[v_1, u]$ and $[v_2, u]$ are two sides of F and u a vertex. As the only non-trivial cyclic subgroups of $PSL(2,\mathbb{R})$ are generated by elliptic elements (with a unique fixed pt in $\mathbb{H}$ each of them) then: There is a one-to-one correspondence between elliptic cycles of F and maximal finite cyclic subgroups of Γ (their conjugacy classes)

<u>Example</u>



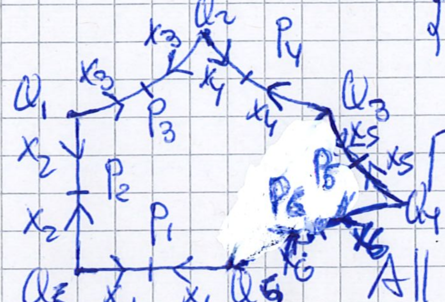
$\Gamma = \langle x, y \mid x^\ell = y^n = (yx)^m = 1 \rangle$

Cycles of vertices $\{P\} \longleftrightarrow [\langle x \rangle]$

$\{Q_1, Q_2\} \longleftrightarrow [\langle y x \rangle]$

$\{R\} \longleftrightarrow [\langle y \rangle]$

In the example we see that the sides of ∂F are identified in pairs; that's always the case. Because if s and s' are sides in F s.t. $\exists\, Id \neq T \in \Gamma$ $s' = T(s)$, then s' side of $T(F)$ and $T(s) = F \cap T(F)$ (<u>Exercise: show the equality</u>).



$\{Q_1, Q_2, Q_3, Q_4, Q_5\}$ angle is 2π.

$\Gamma = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_i^2 = 1 = x_1 x_2 x_3 x_4 x_5 x_6 \rangle$

All surfaces of genus 2 are 2-coverings of these orbifolds. So to study the space of RS of genus 2 one can study such fundamental regions (Klein, Teichmüller, Fox, Näätänen, Kuusalo, ...)

In general one has: that the elements of a Fuchsian group Γ that pair sides of the (Dirichlet) region of Γ generate Γ.

As for lattices and tori one has:

<u>Th</u> Let Γ be a Fuchsian group. Then $\mathbb{H}/\Gamma$ is a R.S and the (branched) covering is a holomorphic map.

In the proof we have to think a bit more only when Γ contains elliptic elements. In this case (Γ with elliptic elements) $\mathbb{H}/\Gamma$ is an orbifold and $p: \mathbb{H} \to \mathbb{H}/\Gamma$ is a branched-covering. But Γ acts properly discontinuously so given $h \in \mathbb{H}$, if $\delta(h) = \inf\{d_H(\gamma h, h)\}_{\gamma \in \Gamma,\ \gamma(h) \neq h} > 0$ and we have an atlas $\{B_h(\delta(h)/2)\}_{h \in \mathbb{H}}$. Observe that if $B_h(h) \cap T(B(h)) \neq \emptyset$ (where we denote $B_h(\delta(h)/2) = B(h)$) then $T(h) = h$. Also $\delta(S(h)) = \delta(h)$ and $B(S(h)) = S B(h)$

A point $q \in \mathbb{H}$ s.t. $B(q) \cap T(Bq) \neq \phi$, then $T = Id$ or q fixed by an elliptic element and the only point in $B(q)$ fixed by an elliptic element is q. The restriction $p_q = p|_{B(q)}$ is continuous and open ~~(and homeomorphism if~~

Let $m(q)$ be the order of the stabilizer of q in Γ.
The map $f_q(z) = \left(\frac{z-q}{z-\bar{q}}\right)^{m(q)}$ analytic (continuous, open)

takes $B(q)$ to ~~an~~ open disc of 0 in $\mathbb{D}$. By Schwartz' Lemma if $T \in \Gamma$, elliptic fixing q, $\frac{T(z)-q}{T(z)-\bar{q}} = \omega\left(\frac{z-q}{z-\bar{q}}\right)$, with ω

an $m(q)$ root of unity, and $f_q(z_1) = f_q(z_2)$ iff $z_2 = T(z_1)$ for some elliptic element $T \in \Gamma$ with $T(q) = q$

Define $\varphi_q = f_q\, p_q^{-1} : p_q(B(q)) \to f_q(B(q)) \subseteq \mathbb{D}$
φ_q is homeo if q not fixed by elliptic element of Γ (as f_q is homeo). As for q being fixed by an elliptic element, we have by the above that $f_q(z_1) = f_q(z_2)$ and $\varphi_q[z] = f_q[z]$, well-defined. Now, if $\varphi_q[z_1] = \varphi_q[z_2]$ then $f_q[z_1] = f_q[z_2]$ and $\exists T'$ elliptic s.t. $z_2 = T'(z_1)$ and $T'(q) = q$, so $[z_1] = [z_2]$. So φ_q homeomorphism.
Moreover the transition function are analytic. In fact, if $p_q(B(q)) \cap p_r(B(r)) \neq \phi$ the function
$$\varphi_r \varphi_q^{-1} = \underbrace{f_r\, p_r^{-1}\, p_q}_{\text{analytic}} f_q^{-1} . \quad \varphi_r \varphi_q^{-1} \text{ analytic except for}$$

possibly f_q^{-1} at 0 if $m(q) > 1$. In these cases $0 \in \varphi_q(p_q(B(q)) \cap p_r(B(r)))$, then q is the only pt in $B(q)$ mapped to 0 by f_q, and $B(q)$ contains no other fixed points by elliptic elements then r, so $r = S(q)$ for some $S \in \Gamma$ so $B(r) = S\,B(q)$ and $p_q(B(q)) = p_r(B(r))$. Again $\varphi_r \varphi_q^{-1}$ analytic.

Th Let Λ_1, Λ_2 be Fuchsian groups without elliptic elements. Then $\mathbb{H}/\Lambda_1$ and $\mathbb{H}/\Lambda_2$ conformally equivalent iff $\exists T \in PSL(2,\mathbb{R})$ s.t. $T\Lambda_1 T^{-1} = \Lambda_2$

Proof we have

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\tilde{g}"] \arrow[d, "p_1"'] & \mathbb{H} \arrow[d, "p_2"] \\ X_1 = \mathbb{H}/\Lambda_1 \arrow[r, "g"] & \mathbb{H}/\Lambda_2 = X_2 \end{tikzcd}$$

p_1, p_2 universal coverings of $\mathbb{H}/\Lambda_1$ and $\mathbb{H}/\Lambda_2$

if $g: X_1 \to X_2$ is conformal homeomorphism, as $\mathbb{H}$ is the universal covering $\exists \tilde{g}: \mathbb{H} \to \mathbb{H}$ where the diagram commutes, $\tilde{g} \in PSL(2,\mathbb{R})$ and $g[z]_{\Lambda_1} = [\tilde{g}(z)]_{\Lambda_2}$
Now for $T \in \Lambda_1$ $[z]_{\Lambda_1} = [Tz]_{\Lambda_1}$ and $[\tilde{g}Tz]_{\Lambda_2} = [\tilde{g}z]_{\Lambda_2}$
So $\exists S \in \Lambda_2$ $\tilde{g}T = S\tilde{g}$ $\forall z \in \mathbb{H}$ So $\tilde{g}T\tilde{g}^{-1} = S$,
$\tilde{g}\Lambda_1\tilde{g}^{-1} \subseteq \Lambda_2$, similarly for $g^{-1}[z]_{\Lambda_2} = [\tilde{g}^{-1}(z)]_{\Lambda_1}$ and
$\tilde{g}^{-1}\Lambda_2\tilde{g} \subseteq \Lambda_1$, so $\tilde{g}\Lambda_2\tilde{g}^{-1} = \Lambda_1$

Th If Λ is a Fuchsian group without elliptic elements then $Aut(\mathbb{H}/\Lambda) = N(\Lambda)/\Lambda$, $N(\Lambda)$ the normalizer in $PSL(2,\mathbb{R})$
Proof Let $t: \mathbb{H}/\Lambda \to \mathbb{H}/\Lambda$ an automorphism, so $t[z]_\Lambda = [Tz]_\Lambda$ for some $T \in PSL(2,\mathbb{R})$ and $\Lambda = T\Lambda T^{-1}$ so $T \in N(\Lambda)$.
Conversely, if $T \in N(\Lambda)$ $t: \mathbb{H}/\Lambda \to \mathbb{H}/\Lambda$ given by $t[z]_\Lambda = [Tz]_\Lambda$ is automorphism. The morphism $T \to t$ has kernel Λ so $Aut(\mathbb{H}/\Lambda) = N(\Lambda)/\Lambda$. $N(\Lambda)$ is also a Fuchsian group
As consequence if Λ is a non-cyclic Fuchsian group without elliptic elements, there is a Fuchsian group Γ s.t. $Aut(\mathbb{H}/\Lambda) = \Gamma/\Lambda$ with $\Lambda \trianglelefteq \Gamma$.

Results

1) If F is a fund region of a Fuchsian group Γ then $\mathbb{H}/\Lambda$ conformally equiv to $F/\sim$ ($\sim$ pairing in ∂F given by Λ)

This is like in the case of $\mathbb{D}/\Omega$ and $F/\sim$
($\sim$ pairing in ∂F by Ω)

2) As a consequence F is compact iff $\mathbb{H}/\Gamma$ is compact

3) $\mathbb{H}/\Gamma$ compact, then Γ contains no parabolic

4) Let F_1 and F_2 two fundamental regions for Γ (where $\mu(\partial F_1) = \mu(\partial F_2) = 0$). Then $\mu(F_1) = \mu(F_2)$.
μ is the h-area of the region in $\mathbb{H}$.

5) Let F be a Dirichlet region of a Fuchsian group Γ. Let $\theta_1, \theta_2, \ldots, \theta_t$ be the internal angles at a congruent set of vertices of F. Let m be the order of the stabilizer in Γ of one of these vertices. Then $\theta_1 + \cdots + \theta_t = \frac{2a}{m}$

6) $\mu(F) = 2a\left[2g - 2 + \sum\left(1 - \frac{1}{m_i}\right)\right]$
(the best place to see this is in Scott's paper)
Again 6 gives us a way of calculating the genus

Th (Poincaré, Maskit) If $g \geq 0$, $m_i \geq 2$ are integers s.t $2g - 2 + \sum\left(1 - \frac{1}{m_i}\right) > 0$ Then there is a Fuchsian group with signature $(g; m_1, \ldots, m_r)$

That is the conjugacy classes of maximal cyclic groups are $[C_{m_1}], \ldots, [C_{m_r}]$ and the surface $\mathbb{H}/\Gamma$ has genus g.
The proof is very tedious and with many cases, both Beardon and Maskit has no complete proofs.
For triangle groups is nice and much easier.

7) As a consequence If F is a Dirichlet region of a

Fuchsian group Γ, $\mu(F) \geq \frac{\pi}{21}$, Equality obtained for triangles groups with signature $(0; 2, 3, 7)$

8) If X is a RS of genus $g \geq 2$ $|Aut(X)| \leq 84(g-1)$

$X = \mathbb{H}/\Lambda$, Λ without elliptic elements $\mu(F_\Lambda) = \mu(X) = 2g-2$

$Aut(X) = N(\Lambda)/\Lambda$ with $N(\Lambda)$ also Fuchsian and cocompact

We have the (branched) covering $X = \mathbb{H}/\Lambda \longrightarrow \mathbb{H}/N(\Lambda)$

so $\mu(F_{N(\Lambda)}) \geq \pi/21$ and $|Aut X| = \frac{\mu(X)}{\mu(F_{N(\Lambda)})} \leq \frac{2\pi 2(g-1)}{\pi/21} = 84(g-1)$

<u>Def</u>. A group of $84(g-1)$ automorphisms of a compact RS of genus g is called a Hurwitz group

<u>Example</u> Kleins Quartic ($y^7 = x(x-1)$) has genus $g=3$ and $168 = 84(3-1)$ automorphisms. So $|Aut(KQ)| = PSL(2,7)$ is a Hurwitz group.

We finish by interpreting 8) as an algorithm (on the monodromies of the orbifolds)

If Λ Fuchsian without elliptic elements $X = \mathbb{H}/\Lambda$ RS where $Aut(X) = N(\Lambda)/\Lambda$ and $Y = \mathbb{H}/N(\Lambda)$ Riemann orbifold, where the (orbifold) covering $X \longrightarrow Y$ is regular, then we have

$\theta : N(\Lambda) \longrightarrow Aut(X)$, the monodromy of the covering

<u>Example</u> $PSL(2,7)$ is generated by one element of order 2 and one of order 7 which product has order 3. So we have $\theta : \Delta(0; 2, 3, 7) \longrightarrow PSL(2,7)$

$\ker \theta = \Lambda$ is Fuchsian without elliptic elements and genus of $\mathbb{H}/\Lambda$ is 3 by Riemann-Hurwitz