Homotopy, Fundamental Group and Coverings

I follow Massey's book (quite loose)

Def Let $f, g : X \xrightarrow{cont} Y$. We say that f and g are homotopic iff $\exists\, H : X \times [0,1] \xrightarrow{cont} Y$ s.t. $\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \end{cases}$ $\forall x \in X$

$H(*, t) : X \xrightarrow{cont} X$, $x \longmapsto H(x,t)$

We write $f \simeq g$ (Notation $I = [0,1]$)

H is a homotopy

Examples i) $Id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \longmapsto x$; $0 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \longmapsto 0$

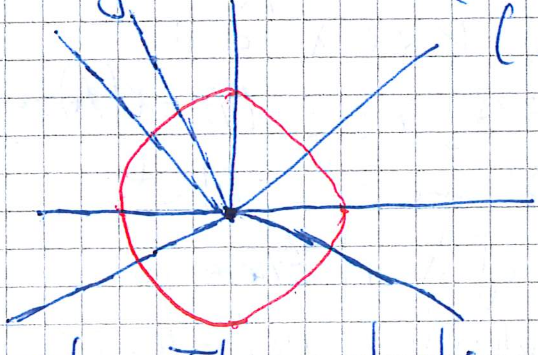
with homotopy $H : \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n$

$(x,t) \longmapsto H(x,t) = (1-t)x + t0$

ii) $Id : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$, $x \longmapsto x$, and $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$, $x \longmapsto \frac{x}{|x|}$

Again $H : (\mathbb{R}^n \setminus \{0\}) \times I \longrightarrow \mathbb{R}^n$

$(x,t) \longmapsto H(x,t) = (1-t)x + t\frac{x}{|x|}$



Obs The relation on $\mathcal{C}(X,Y)$ given by $f \simeq g$ is an equivalence relation

Exercise Verify it!

Obs If $f_1 \simeq g_1$ and $f_2 \simeq g_2$ with $f_1, g_1 : X \to Y$ and $f_2, g_2 : Y \to Z$, then $f_2 \circ f_1 \simeq g_2 \circ g_1$ with homotopy composition $H_1 : X \times I \to Y$ and $H_2 : Y \times I \to Z$

$\begin{cases} H_1(x,0) = f_1(x), H_1(x,1) = g_1(x) \\ H_2(y,0) = f_2(y), H_2(y,1) = g_2(y) \end{cases}$

$f_2 \circ H_1$ homotopy $f_2 \circ f_1 \simeq f_2 \circ g_1$ and $H_2 \circ (g_1 \times Id_I)$ takes $f_2 \circ g_1 \simeq g_2 \circ g_1$ because of the transitivity $f_2 \circ f_1 \simeq g_2 \circ g_1$

Def We say that a space X is contractible if $1_d : X \longrightarrow X$ is homotopic to a constant map $c : X \longrightarrow X$.

We have seen that $\mathbb{R}^n$ with the usual topology is contractible.

Clearly a space X is contractible iff given two continuous maps $f, g : X \to X$; $f \simeq g$ since $1_d \circ f \simeq c \circ f \simeq c \circ g \simeq 1_d \circ g$

Def i) Two spaces X, Y are homotopy equivalent if there exist $f : X \xrightarrow{cont} Y$ and $g : Y \xrightarrow{cont} X$ such that $g \circ f \simeq 1_{d_X}$ and $f \circ g \simeq 1_{d_Y}$, f and g are called homotopy equivalences.

ii) Let X be a space and A a subspace of X we say that two maps $f, g : (X, A) \xrightarrow{cont} (X, B)$ are homotopic relative to A if there exists a homotopy $H : X \times I \xrightarrow{cont} X$ s.t $H(x,0) = f(x)$, $H(x,1) = g(x)$ $\forall x \in X$ and
$H(a,t) = f(a) = g(a)$, $a \in A$, $t \in I$ $(B = f(A) = g(A))$

iii) Given a space X and $A \subseteq X$ we say that A is a deformation retract of X if there is a retraction $r : X \xrightarrow{cont} A$ which is homotopic to the identity of X relative to A. This is equivalent to say that $r \circ i \simeq 1_{d_A}$, and $i \circ r \simeq 1_{d_X}$

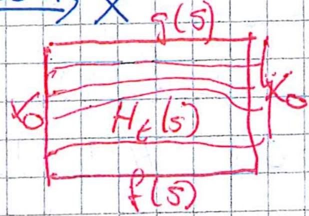
Notice that the homotopies in Examples i) and ii) satisfy
$$\begin{cases} \text{i) } H(0,t) = 0 \quad \forall t \in I \\ \text{ii) if } |x| = 1 \quad H(x,t) = x - tx + tx = x, \ \forall t \end{cases}$$

so we have seen that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$, with retraction $r : \mathbb{R}^n \setminus \{0\} \to S^1$ $r(x) = \frac{x}{|x|}$ and $\{0\}$ is a deformation retract of $\mathbb{R}^n$

Def i) A loop based on x_0 is a closed path $f: I \xrightarrow{cont} X$ (no other intersection than) $f(0)=f(1)=x_0$

ii) Two loops f, g are homotopic if $f \simeq g$ relative to $\{x_0\}$
i.e there exists $H: \cancel{X} \times I \xrightarrow{cont} X$

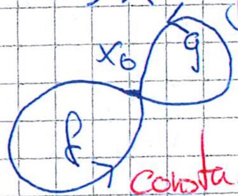
$H(0,t)=x_0$, $H(1,t)=x_0$
$H(s,0)=f(s)$ $H(s,1)=g(s)$



We can define the product of two loops $f*g$

$f: I \xrightarrow{cont} X$ $f(0)=f(1)=x_0$ as $f*g: I \longrightarrow X$, $s \longmapsto \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ f(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$

$g: I \xrightarrow{cont} X$ $g(0)=g(1)=x_0$



If $e: I \xrightarrow{\text{constant loop}} X$, $s \longmapsto e(s)=x_0$ we have that if $f: I \xrightarrow{cont} X$ loop $f(0)=f(1)=x_0$

$f*e = f$ and $e*f = e$

The symmetric loop of a loop f, $\bar{f}$ is defined by
$\bar{f}: I \longrightarrow X$ $\bar{f}(s) = f(1-s)$; $\bar{f}(0) = f(1) = f(0) = \bar{f}(1) = x_0$

Observe that $\bar{f} * f \neq e$

$(\bar{f} * f)(1/4) = f(1/2) \neq x_0$

Def We say that a space X (are connected) is simply connected, or 1-connected, if each loop $f: I \longrightarrow X$, $f(0)=f(1)=x$ is homotopic to ~~the~~ the constant loop $e_x: I \to X$, $e(s)=x$

Def We denote by $[f]$ the homotopy class of a loop based at $x_0 \in X$, X a topological space (connected)

We denote by $\Omega(X, x_0)$ the space of loops based on x_0. Let $\pi_1(X, x_0) = \{[f] ; f \in \Omega(X, x_0)\}$ denote the fundamental group of X at x_0

Th: $\pi_1(X, x_0)$ is a group with $[f]\cdot[g] = [f*g]$

1) First, we show that the product defined above is well defined. If $f_1 \simeq_{x_0} f_2$, $g_1 \simeq_{x_0} g_2$, then $f_1 * g_1 \simeq_{x_0} f_2 * g_2$

We have $H_1 : I \times I \xrightarrow{\text{cont}} X$; $H_2 : I \times I \xrightarrow{\text{cont}} X$

$H_1(s,0) = f_1(s)$; $H_1(s,1) = f_2(s)$; $H_2(s,0) = g_1(s)$; $H_2(s,1) = g_2(s)$

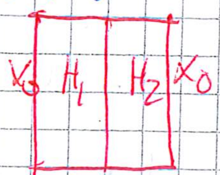
$H_1(0,t) = H_1(1,t) = x_0$; $H_2(0,t) = H_2(1,t) = x_0$

$$H : I \times I \xrightarrow{\text{cont}} X$$
$$(s,t) \longmapsto \begin{cases} H_1(2s,t) & 0 \le s \le \frac{1}{2} \quad \text{cont } [0,\tfrac{1}{2}]\times I \\ H_2(2s-1,t) & \frac{1}{2} \le s \le 1 \quad \text{cont } [\tfrac{1}{2},1]\times I \end{cases}$$

$$H(s,0) = \begin{cases} H_1(2s,0) = f_1(2s) & 0 \le s \le \frac{1}{2} \\ H_2(2s-1,0) = g_1(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} = f_1 * g_1(s,0)$$

$$H(s,1) = \begin{cases} H_1(2s,1) = f_2(2s) \\ H_2(2s-1,1) = g_2(2s-1) \end{cases} = f_2 * g_2(s,1)$$

$H(0,t) = H_1(0,t) = x_0$; $H(1,t) = H_2(1,t) = x_0$

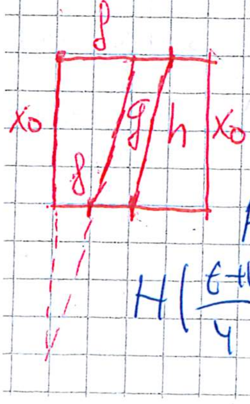


2) Product is associative

$$(f*g)*h = \begin{cases} f(4s), & 0 \le s \le \frac{1}{4} \\ g(4s-1), & \frac{1}{4} \le s \le \frac{1}{2} \\ h(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$f*(g*h) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(4s-2), & \frac{1}{2} \le s \le \frac{3}{4} \\ h(4s-3) & \frac{3}{4} \le s \le 1 \end{cases}$$

$$H(s,t) = \begin{cases} f\left(\frac{4s}{t+1}\right), & 0 \le t \le 1, \ 0 \le s \le \frac{t+1}{4} \\ g(4s-(t+1)) & 0 \le t \le 1, \ \frac{t+1}{4} \le s \le \frac{t+2}{4} \\ h\left(1 - \frac{4(1-s)}{2-t}\right) & 0 \le t \le 1, \ \frac{t+2}{4} \le s \le 1 \end{cases}$$
continuous



$$H(s,0) = \begin{cases} f(4s) & 0 \le s \le \frac{1}{4} \\ g(4s-1) & \frac{1}{4} \le s \le \frac{1}{2} \\ h(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} \qquad H(s,1) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(4s-2) & \frac{1}{2} \le s \le \frac{3}{4} \\ h(4s-3), & \frac{3}{4} \le s \le 1 \end{cases}$$

$H(0,t) = f(0) = x_0$; $H(1,t) = h(1) = x_0$

$H\left(\frac{t+1}{4}, t\right) = f(1) = g(0) = x_0$; $H\left(\frac{t+2}{4}, t\right) = g(1) = h(0) = x_0$

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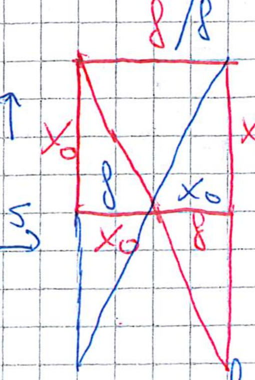
3) There exists <u>identity</u> (trivial element)

Consider $e: I \longrightarrow X$, $s \longmapsto x_0$. To see that $\begin{cases} [e][f]=[f] \\ [f][e]=[f] \end{cases}$



$$H_e(s,t) = \begin{cases} x_0 & 0 \le t \le 1,\ 0 \le s \le \frac{1-t}{2} \\ f\left(1 - \frac{2(1-s)}{t+1}\right) & 0 \le t \le 1,\ \frac{1-t}{2} \le s \le 1 \end{cases}$$

cont

$[e]*[f] \simeq [f]$

$$H_r(s,t) = \begin{cases} f\left(\frac{2s}{t+1}\right), & 0 \le t \le 1,\ 0 \le s \le \frac{t+1}{2} \\ x_0, & 0 \le t \le 1,\ \frac{1+t}{2} \le s \le 1 \end{cases}$$

cont

$f * e \simeq_{x_0} f$

Exercise: Control that $\begin{cases} H_e(0,t) = H_e(1,t) = x_0 \\ H_r(0,t) = H_r(1,t) = x_0 \\ H_e\left(\frac{1-t}{2}, t\right),\ H_r\left(\frac{1+t}{2}, t\right) \text{ well defined} \end{cases}$

4) Existence of <u>inverse</u>

Let $f: I \xrightarrow{\text{cont}} X$, $s \longmapsto f(s)$ loop at x_0, $f(0) = f(1) = x_0$ and consider

$\bar{f}: I \xrightarrow{\text{cont}} X$, $s \longmapsto f(1-s)$ loop at x_0, $\bar{f}(0) = \bar{f}(1) = x_0$

First of all if $f \simeq_{x_0} g$ then $\bar{f} \simeq_{x_0} \bar{g}$; if $H: I \times I \longrightarrow X$, (s,t) is the homotopy relative to x_0

$H(s,t) \ni \begin{cases} H(s,0) = f(s) \\ H(s,1) = g(s) \end{cases}$, $H(0,t) = H(1,t) = x_0$, then

$\bar{H}(s,t) = H(1-s,t)$ satisfies $\begin{cases} \bar{H}(s,0) = H(1-s,0) = \bar{f}(s) \\ \bar{H}(s,1) = H(1-s,1) = \bar{g}(s) \end{cases}$

$\bar{H}(0,t) = H(1,t) = x_0 = H(0,t) = \bar{H}(1,t)$

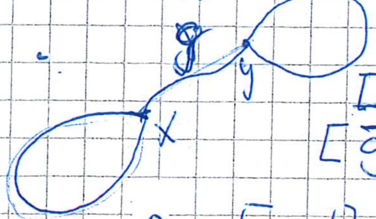
<u>Examples</u> i) $X = \{x_0\}$; $\pi_1(X, x_0) = \{e\}$

ii) X contractible given $x \in X$, any loop f based at x; $f \simeq_x e$

So $\forall x \in X$; $\pi_1(X, x) = \{e\}$

iii) X simply-connected, given $x \in X$, again any loop based at x; $f \simeq_x e$ so, $\forall x \in X$, $\pi_1(X, x) = \{e\}$

Th If X is arc-connected, then $\pi_1(X,x)$ and $\pi_1(X,y)$ are isomorphic for any two points $x,y\in X$



We define $g_*: \pi_1(X,x) \longrightarrow \pi_1(X,y)$

$$\alpha \longmapsto \gamma^{-1}\alpha\gamma$$

$[g]=\gamma$

$[\bar{g}]=\gamma^{-1}$

$\gamma^{-1}\alpha\gamma = g_*(\alpha)$

$g:[0,1] \xrightarrow{\text{cont}} X$, $g(0)=x$, $g(1)=y$; $\bar{g}:[0,1]\longrightarrow X$, $\bar{g}(s)=g(1-s)$ $\begin{cases}\bar{g}(0)=y\\ \bar{g}(1)=x\end{cases}$

$[g]=\{f: I\longrightarrow X,\ f(0)=x,\ f(1)=y,\ f\simeq_{\{x,y\}} g\}$

We have $\bar{g}_*: \pi_1(X,y)\longrightarrow \pi_1(X,x)$

$$\beta \longmapsto \gamma\beta\gamma^{-1}$$

Observe that $\bar{g}_*$ and g_* are inverse to each other

g_* (and $\bar{g}_*$) homomorphism since

$$g_*(\alpha_1\alpha_2)=\gamma^{-1}\alpha_1\alpha_2\gamma=\gamma^{-1}\alpha_1\gamma\gamma^{-1}\alpha_2\gamma=g_*(\alpha_1)g_*(\alpha_2)$$

Now, as if $f: I\longrightarrow X$ is another path from x to y

$f\simeq_{x,y} g$ iff $f\bar{g}\simeq_x x$ (and $\bar{f}g\simeq_y y$) we have

that f_* and g_* are the same isomorphism if

$f\simeq_{xy} g$ homotopic relative to $\{x,y\}$.

Now let $\varphi: X\longrightarrow Y$ a continuous map and let

f,g loops based at x_0 let $[f]=\alpha$, $[g]=\beta$ be their

homotopy classes. φ induces a homomorphism

$\varphi_*: \pi_1(X,x_0)\longrightarrow \pi_1(Y,\varphi x_0)$. Observe that

$\varphi_*(e_{x_0})=e_{\varphi x_0}$, that $\varphi_*(\alpha\beta)=\varphi_*(\alpha)\varphi_*(\beta)$, $\varphi_*(\alpha^{-1})=(\varphi_*\alpha)^{-1}$

It is well-defined because $f\simeq_{x_0} g$ then $\varphi\circ f\simeq_{\varphi x_0}\varphi\circ g$.

Observe that if φ is homeomorphism then φ_* is an isomorphism

Now if A is a deformation retract of X
$i:(A,a) \longrightarrow (X,a)$, and $r:(X,a) \xrightarrow{\text{cont}} (A,a)$
$a \in A$
$i \circ r \simeq_A \mathrm{Id}_X$, and $r \circ i \simeq_A \mathrm{Id}_A$ so $i_* \circ r_* \underset{\text{Id}}{=} \pi_1(X,a) \to \pi_1(X,a)$
$\underset{\text{Id}}{r_* i_*} : \pi_1(A,a) \longrightarrow \pi_1(A,a)$ so i_* and r_*
isomorphisms $r_*: \pi_1(X,a) \longrightarrow \pi_1(A,a)$
$i_*: \pi_1(A,a) \longrightarrow \pi_1(X,a)$
As example given x with $|x|=1$ we have that
$\pi_1(\mathbb{R}^n \setminus \{0\}, x)$ isomorphic to $\pi_1(S^{n-1}, x)$ but
$\mathbb{R}^n \setminus \{0\}$ is not homeomorphic to S^{n-1}
Clearly if a space X is contractible (to a point), it is simply-connected.
The Rotation Index Theorem tells us that
$\pi_1(S^1, x) = \mathbb{Z}$. Then $\pi_1(\mathbb{R}^2 \setminus \{0\}, x) = \mathbb{Z}$.

Example i) Consider an open ball $B^n(x,r)$, $r>0$
As $B^n(x,r)$ homeomorphic to $\mathbb{R}^n$ we know that
$\pi_1(B^n(x,r), x)$ isomorphic to $\pi_1(\mathbb{R}^n, x) = \{e_x\}$
ii) We also know that $\pi_1(S^n, x) = \{e_x\}$ for $n \geq 2$
since $S^2 \underset{\text{homeo}}{\simeq} \mathbb{C}_S \cup \mathbb{C}_N$ where $\mathbb{C}_S$ is the complex plane seen from the North Pole N, and $\mathbb{C}_N$ is the complex plane seen from the South Pole
$\mathbb{C}_S \cap \mathbb{C}_N \equiv \mathbb{C} \setminus \{0\}$ and given $z \in \mathbb{C}_S \cap \mathbb{C}_N$
$\pi_1(\mathbb{C}_S, z) = \pi_1(\mathbb{C}_N, z) = \{e_z\}$. Using the Lebesgue number of the open covering $\{f^{-1}(\mathbb{C}_S), f^{-1}(\mathbb{C}_N)\}$ of $I=[0,1]$ induced by the loop $f: I \longrightarrow S^n$, $f(0)=f(1)=z$
we obtain that $\pi_1(S^2, z) = \{e_z\}$. As there is a homeomorphism of S^2 taking z to N (or S) $\pi_1(S^2, N) = \{e_N\}$

iii) Using homology we can prove that S^{n-1} is not a deformation retract of $\mathbb{R}^n$. If that it is the case then $S^{n-1} \simeq \mathbb{R}^n$ and they will have the same homology which is NOT the case. Bearing in mind this we can prove Brouwer fixed-point theorem:
Given the closed unit disc $D^n = \{x \in \mathbb{R}^n ; |x| \le 1\}$. Any continuous map $f: D^n \longrightarrow D^n$ has at least a fixed point. Otherwise if $\forall x \in D^n$; $f(x) \neq x$
Consider the ray $\overrightarrow{f(x)x} = R_x$; $R_x \cap S^{n-1} = \{y\}$
In fact $y - x = \lambda (x - f(x))$ with $\lambda > 0$. The map $r: D^n \longrightarrow S^{n-1}$ given by $r(x) = y$ is a retraction with $r(x) = x$ if $x \in S^{n-1}$. But this is impossible so there is a point x_0 s.t

$$f(x_0) = x_0$$

Th The fundamental group of the product space $\pi_1(X \times Y, (x,y))$ is <u>naturally</u> isomorphic to the direct product of fundamental groups $\pi_1(X,x) \times \pi_1(Y,y)$
The isomorphism is defined by the map a_*
$\alpha \in \pi_1(X \times Y, (x,y)) \longmapsto (p_*(\alpha), q_*(\alpha))$ where $p: X \times Y \longrightarrow X$ and $q: X \times Y \longrightarrow Y$ are the (continuous) projections of the product space onto its factors
Non proof but. The idea of the proof is that a continuous function $f: I \longrightarrow X \times Y$ is a loop at (x,y) <u>iff</u> $p \circ f$ and $q \circ f$ are loops in X based at x respectively in Y at y. Here $f \underset{(x,y)}{\simeq} g$ iff $p \circ f \underset{X}{\simeq} p \circ g$ and $q \circ f \underset{Y}{\simeq} q \circ g$. Moreover if $h = f * g$ then $p \circ h = (p \circ f) * (p \circ g)$ and $q \circ h = (q \circ f) * (q \circ g)$. Finish the proof by seeing that a_* well-defined and isomorphism.

-Conversely: fix $x_0 \in X$, $y_0 \in Y$ then the continuous maps $i: X \to X \times Y$, $x \mapsto (x, y_0)$ and $j: Y \to X \times Y$, $y \mapsto (x_0, y)$ define mono homomorphisms $i_*: \pi_1(X, x_0) \to \pi_1(X \times Y, (x_0, y_0))$ and $j_*: \pi_1(Y, y_0) \to \pi_1(X \times Y (x_0, y_0))$. It is easy to prove that $b_*: \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$ given by $b_*(\beta, \gamma) = i_*(\beta)\, j_*(\gamma)$ is the inverse isomorphism ~~to~~ a_*.

Example i) The torus $\mathbb{T} = S^1 \times S^1$, so $\pi_1(\mathbb{T}, -) = \mathbb{Z} \times \mathbb{Z}$
(as the torus is arc-connected any point will give the same group)

ii) $\pi_1(S^1 \times \{y_0\}, (x_0, y_0)) = \mathbb{Z}$. Then $S^1 \times \{y_0\}$ cannot be a deformation retract of the torus

We finish this section by stating that if X and Y are homotopy equivalent, i.e there are maps $f: X \xrightarrow{cont} Y$, $g: Y \xrightarrow{cont} X$ s.t $g \circ f = 1d_X$ and $f \circ g = 1d_Y$ then

$$\pi_1(X, x) \overset{\text{isomorphic}}{\cong} \pi_1(Y, f(x))$$

Consider h_0 and $h_1: X \xrightarrow{cont} Y$ and homotopic with homotopy $h: X \times I \to Y$. For $x_0 \in X$

$h_{0*}: \pi_1(X, x_0) \to \pi_1(Y, h_0(x_0))$
$h_{1*}: \pi_1(X, x_0) \to \pi_1(Y, h_1(x_0))$ and the homotopy

φ induces an isomorphism $u_Y: \pi_1(Y, h_0(x_0)) \to \pi_1(Y, h_1(x_0))$ given by $\alpha \in \pi_1(Y, h_0(x_0))$ $u_Y(\alpha) = \gamma^{-1} \alpha \gamma$ with γ the homotopy class of the path $t \mapsto h(x_0, t)$, $0 \le t \le 1$

The diagrams

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "u_X"'] & \pi_1(Y, f(x_0)) \arrow[d, "g_*"] \arrow[dr, "u_Y"] & \\
& \pi_1(X, g \circ f(x_0)) \arrow[r] & \pi_1(Y, f \circ g \circ f(x_0))
\end{tikzcd}$$

Making g_* (and f_*) an isomorphism

Seifert and Van Kampen Th

Let X be an arc-connected space such that $X = U \cup V$ with U, V arc-connected subspaces and $U \cap V$ arc-connected. We have

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[dr, "U_*"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_{U*}"] \arrow[rr] \arrow[dr, "i_{V*}"'] & & \pi_1(X, x_0) \\
 & \pi_1(V, x_0) \arrow[ur, "V_*"'] &
\end{tikzcd}$$

$x_0 \in U \cap V$

Where i_{U*}, i_{V*}, U_*, V_* are the homomorphisms induce by the corresponding inclusions.
Seifert and Van Kampen proved (independently) that $\pi_1(X, x_0)$ is the "freeest" possible product of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ (that all the amalgamation is contained in $\pi_1(U \cap V, x_0)$)

Th Let X, U, V, x_0 as above. Let H any group, ρ_1, ρ_2, ρ_3 morphisms s.t the diagram is commutative

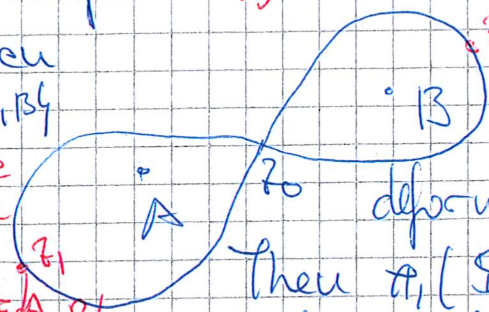
$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[dr, "\rho_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_{U*}"] \arrow[rr, "\rho_3"] \arrow[dr, "i_{V*}"'] & & H \\
 & \pi_1(V, x_0) \arrow[ur, "\rho_2"'] &
\end{tikzcd}$$

Then there exists a unique morphism $\rho : X \to H$ s.t the following three diagrams are commutative

$$\begin{tikzcd}
\pi_1(U) \arrow[r, "U_*"] \arrow[dr, "\rho_1"'] & \pi_1(X) \arrow[d, "\rho"] \\
 & H
\end{tikzcd}
;
\begin{tikzcd}
\pi_1(V) \arrow[r, "V_*"] \arrow[dr, "\rho_2"'] & \pi_1(X) \arrow[d, "\rho"] \\
 & H
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[dr, "\rho_3"'] & \pi_1(X) \arrow[d, "\rho"] \\
 & H
\end{tikzcd}$$

With other words $\pi_1(X)$ is the product of $\pi_1(U)$ and $\pi_1(V)$ amalgamated at the common subgroup $i_{U*}(\pi_1(U \cap V))$ identify with $i_{V*}(\pi_1(U \cap V))$

Examples i) Consider $S^2 \setminus \{A,B,\infty\} \cong \mathbb{C}\setminus\{A,B\}$



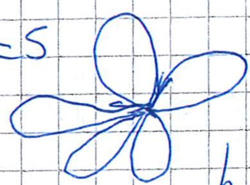
Then $\mathbb{C}\setminus\{A,B\}$ We can find X the union of two circles which is a deformation retract of $\mathbb{C}\setminus\{A,B\}$

We have to base the loops at $z_0 \notin \{A,B\}$

Then $\pi_1(S^2\setminus\{A,B,\infty\}) = \pi_1(\mathbb{C}\setminus\{A,B\}) = \pi_1(X)$
and by Seifert-Van Kampen $\pi_1(X) = \mathbb{Z} * \mathbb{Z} = F_2$
Take $U = X\setminus\{z_1\}$ and $V = X\setminus\{z_2\}$ $\pi_1(U)=\pi_1(V)=\mathbb{Z}$
$\pi_1(U\cap V) = \pi_1(\{z_0\}) = \{e\}$ $U\cap V$ contractible to $\{z_0\}$

ii) Let X a bouquet of n circles (with common intersection just at z_0). Then $\pi_1(X) = F_n$

$n=5$



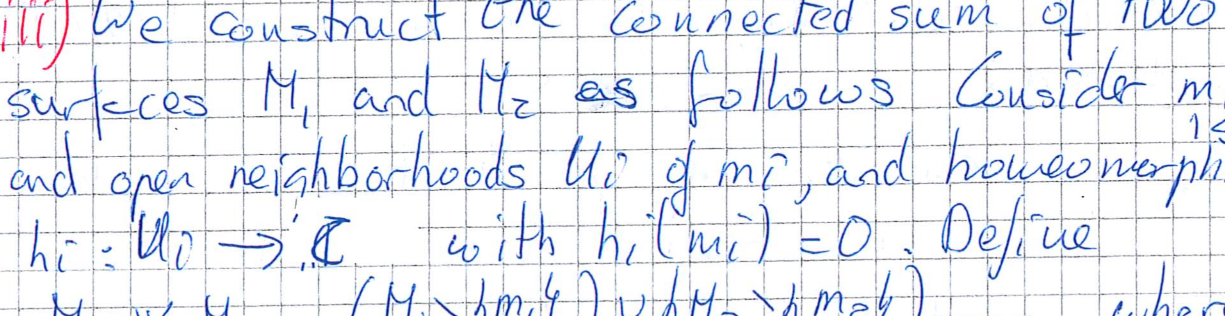
F_n is the free group on n generators
In the same way $\hat{\mathbb{C}}\setminus\{n+1 \text{ pts}\} \cong \mathbb{C}\setminus\{n \text{ pts}\}$
has $\pi_1(\hat{\mathbb{C}}\setminus\{n+1 \text{ pts}\}) = F_n$.

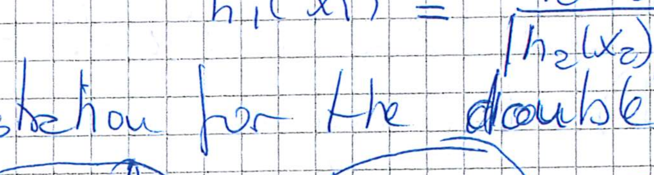
iii) We construct the connected sum of two surfaces M_1 and M_2 as follows Consider $m_i \in M_i$, $1\le i\le 2$
and open neighborhoods U_i of m_i, and homeomorphisms $h_i : U_i \to \mathbb{C}$ with $h_i(m_i) = 0$. Define

$$M_1 \# M_2 = \frac{(M_1\setminus\{m_1\}) \cup (M_2\setminus\{m_2\})}{\sim}$$ where

Given $x_1 \in U_1\setminus\{m_1\}$ and $x_2 \in U_2\setminus\{m_2\}$ $x_1 \sim x_2$ iff
$$h_1(x_1) = \frac{h_2(x_2)}{|h_2(x_2)|^2}$$
Then $\pi_1(M_1\#M_2) = \pi_1(M_1) * \pi_1(M_2)$

Illustration for the double torus





According to Seifert Van Kampen and as $\pi_1(U_1) = \pi_1(U_2) = \{e\}$

$\pi_1(T_1 \# T_2) = \langle a_1, b_1, a_2, b_2 ; a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle$

In general $\pi_1(T_1 \# \cdots \# T_g) =$

$= \langle a_1, b_1, \dots a_g, b_g ; \prod_{i=1} [a_i, b_i] = 1 \rangle$

$[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$, the commutator of a_i and b_i

As a consequence let S_g a closed orientable Riemann surface of genus g, S_g is the connected sum of g tori and $\pi_1(S_g)$ as above. If we consider the abelianited quotient group of $\pi_1(S_g)$ e

$H = \pi_1(S_g) / [\pi_1(S_g), \pi_1(S_g)]$; we impose the relations that $[a, b] = 1$

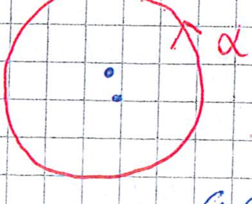
We get that $\forall i$ $[a_i, b_i] = 1$ and so

$H = \langle a_1, b_1, \dots a_g, b_g ; [a_i, b_i] = 1 \rangle = (\mathbb{Z} \times \mathbb{Z}) \times \cdots \times (\mathbb{Z} \times \mathbb{Z})$

$= \mathbb{Z}^{2g} = H_1(S_g)$

ii) Consider $\mathbb{C} \setminus \{0, \infty\}$; $\pi_1(\mathbb{C} \setminus \{0, \infty\}) = \pi_1(\mathbb{C} \setminus \{0\})$

a the class of α $= \mathbb{Z} = \langle a \rangle$



consider the quotient space L_n given by the orbits of the rotation around 0 (in $\mathbb{C}$) with angle $2\pi/n$; we get a new relation $a^n = 1$ So the fundamental group of the quotient is $\pi_1(L_n) = \langle a ; a^n = 1 \rangle = C_n$

The completion, also called L_n is $\mathbb{C}$ with two singular pts (ramified) at 0 and ∞

Observe that $f(z) = z^n$ has a 0 of multip. n at 0 and a pole of multipl. n at ∞