The Gauss - Bonnet Theorem

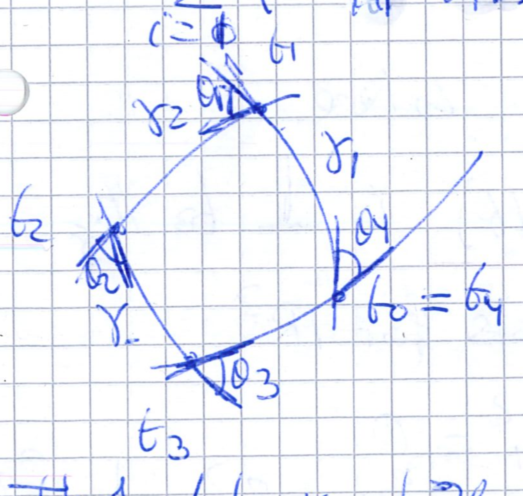
Def A curve $\gamma(t) = x(x_1(t), x_2(t))$ on a (chart of a) surface is called a simple closed piecewise regular if $\gamma(0) = \gamma(l)$, l period, $\gamma(t)$ is simple and there exist $0 = t_0 < t_1 < \dots < t_{k-1} < t_{k} = l$ such that γ regular in each $[t_i, t_{i+1}]$, $0 \le i \le k-1$, $\gamma(t_i)$ vertices. Moreover $\forall t_i$ $\lim\limits_{t \to t_i} \dot\gamma(t) = \dot\gamma(t_i - 0)$ and $\lim\limits_{t \searrow t_i} \dot\gamma(t) = \dot\gamma(t_i + 0)$

Let θ_i be the smallest determination of the angle from $\dot\gamma(t_i - 0)$ to $\dot\gamma(t_i + 0)$, θ_i is the external angle

Consider the functions $\varphi_{i+1} : [t_i ; t_{i+1}] \to \mathbb{R}$ that measures the total variation of the angle at each regular arc

Theorem (of Turning Tangents or Hopf's Umlaufsatz)

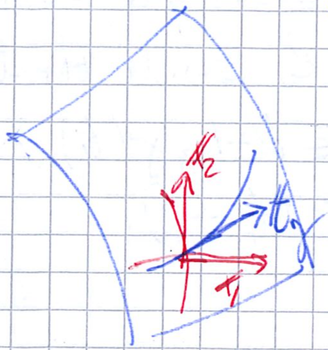
$$\sum_{i=0}^{k-1} \left(\varphi_{i+1}(t_{i+1}) - \varphi_{i+1}(t_i) \right) + \sum \theta_i = 2\pi$$



Th 1 Let $x = (U, x, W)$ be a chart in polar geodesic coordinates for an oriented surface (S, N) and let $\gamma = x(x_1(t), x_2(t))$ be a curve in x.

Assume that $\{X_1(t), X_2(t), N(t)\}$ is positive of the space. Let δ be the angle formed by $t_\gamma(t)$ an $X_1(t)$: Then

$$k_g(\gamma(t)) = \frac{d\delta}{dt} + \frac{\partial\sqrt{g_{22}}}{\partial x_1}\frac{dx_2}{dt}$$



Proof. We can express

$$t_\gamma(t) = \cos\delta(t)\, X_1(t) + \frac{1}{\sqrt{g_{22}(t)}}\sin\delta(t)\, X_2(t)$$

Then $k(\gamma(t))\, n_\gamma(t) =$

$$= -\delta'(t)\sin\delta(t)\, X_1(t) + \cos\delta(t)\frac{dX_1(t)}{dt} +$$

$$+ \delta'\cos\delta(t)\frac{1}{\sqrt{g_{22}(t)}}X_2(t) + \sin\delta(t)\left(\frac{d\frac{1}{\sqrt{g_{22}(t)}}}{dt}\right)X_2(t)$$

$$+ \frac{\sin\delta(t)}{\sqrt{g_{22}(t)}}\frac{dX_2(t)}{dt}$$

To study $k_g(\gamma(t))$ we must project on the tangent plane

Projecting the terms in $\boxed{}$ yields

$$\delta'\left(-\sin\delta(t)\, X_1 + \frac{\cos\delta(t)}{\sqrt{g_{22}(t)}}X_2(t)\right) \quad \text{where}$$

$w(t) = -\sin\delta(t)\, X_1 + \frac{\cos\delta(t)}{\sqrt{g_{22}(t)}}X_2(t)$ is $\perp$ to t_γ

and so $\{t_\gamma, w, N\}$ is ON-basis for $\mathbb{R}^3$

We know that $\Gamma_{11}^1 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{11}^2 = 0$

$\Gamma_{22}^1 = -\frac{1}{2}\frac{\partial g_{22}}{\partial x_1}$, $\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x_1}$, $\Gamma_{22}^2 = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x_2}$

so $\cos\delta\frac{dX_1(t)}{dt} = \cos\delta\left(X_{11}\frac{dx_1}{dt} + X_{12}\frac{dx_2}{dt}\right)$

$$= \cos\delta\left(L_{11}\frac{dx_1}{dt}N + \left(\frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x_1}X_2 + L_{12}N\right)\frac{dx_2}{dt}\right)$$

Projecting we obtain

$$\cos\sigma \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x_1} \frac{dx_2}{dt} X_2 = \frac{1}{2\sqrt{g_{22}}} \frac{\partial g_{22}}{\partial x_1} \frac{dx_2}{dt} \left(\cos\sigma \frac{1}{\sqrt{g_{22}}} X_2\right)$$

In the same way $\frac{1}{\sqrt{g_{22}}} \sin\sigma(t) \frac{dx_2}{dt}$ gets

$$\frac{1}{\sqrt{g_{22}}} \sin\sigma(t) \left[\frac{dx_1}{dt} \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x_1} X_2 + \frac{dx_2}{dt} \left(-\frac{1}{2} \frac{\partial g_{22}}{\partial x_1} X_1 + \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x_2} X_2\right)\right]$$

The term $\frac{d\left(\frac{1}{\sqrt{g_{22}}}\right)}{dt} \sin\sigma \, X_2$ equals

$$-\frac{1}{2g_{22}} \frac{1}{\sqrt{g_{22}}} \left(\frac{\partial g_{22}}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial g_{22}}{\partial x_2} \frac{dx_2}{dt}\right) \sin\sigma \, X_2$$

Adding $\frac{1}{\sqrt{g_{22}}} \sin\sigma \left(-\frac{1}{2\sqrt{g_{22}}} \frac{\partial g_{22}}{\partial x_1} \frac{dx_2}{dt}\right) X_1$

Adding all the projections of $k(\gamma(t)) n_\gamma(t)$ is

$$\left(\frac{d\sigma}{dt} + \frac{1}{2\sqrt{g_{22}}} \frac{\partial g_{22}}{\partial x_1}\right) \vec{W}$$

$$k(\gamma(t)) n_\gamma(t) = \left(\frac{d\sigma}{dt} + \frac{1}{2\sqrt{g_{22}}} \frac{\partial g_{22}}{\partial x_1}\right) \vec{W} + k_n(\gamma(t)) N(t)$$

$$\boxed{\text{So } k_g(\gamma(t)) = \frac{d\sigma}{dt} + \frac{1}{2\sqrt{g_{22}}} \frac{\partial g_{22}}{\partial x_1}}$$

We can prove now

Th (Gauss-Bonnet Theorem). Consider a simple closed piecewise regular curve with regular arcs γ_i and external angles $\theta_1, \dots, \theta_n$ at the vertices $\gamma(t_0), \dots, \gamma(t_n)$ and a chart $X = (\mathcal{U}, x, \omega)$ with $\mathcal{U}$ homeomorphic to a disc

Then

$$\boxed{\sum_{0}^{n}\int_{t_i}^{t_{i+1}} k_g(\gamma_i(s))\,ds + \iint_R K\,d\sigma + \sum_{i=0}^{n}\theta_i = 2\pi}$$

Proof. Assume that the polygon is inside a polar geodesic chart

$$k_g(\gamma_i) = \frac{d\varphi_i}{ds} + \frac{\partial\sqrt{g_{22}}}{\partial x_1}\frac{dx_2}{ds}$$

Integrating $\int_{\gamma_i} k_g(\gamma_i) = \int_{\gamma_i}\frac{d\varphi_i}{ds}\,ds + \int_{\gamma_i}\frac{\partial\sqrt{g_{22}}}{\partial x_1}\frac{dx_2}{ds}\,ds$

$$= \varphi_i(t_{i+1}) - \varphi_i(t_i)$$

$$\sum k_g(\gamma_i) = 2\pi - \sum\theta_i$$

Besides $\sum\int_{\gamma_i}\frac{\partial\sqrt{g_{22}}}{\partial x_1}\frac{dx_2}{ds}\,ds = \sum\int_{\gamma_i}\frac{\partial\sqrt{g_{22}}}{\partial x_1}\,dx_2$

$$\overset{\text{stokes}}{=} \iint_R d\left(\frac{\partial\sqrt{g_{22}}}{\partial x_1}\,dx_2\right)$$

$$d\left(\frac{1}{2\sqrt{g_{22}}}\frac{\partial g_{22}}{\partial x_1}\,dx_2\right) = \left[-\frac{1}{4}\frac{1}{g_{22}}\frac{1}{\sqrt{g_{22}}}\left(\frac{\partial g_{22}}{\partial x_1}\right)^2 + \frac{1}{2}\frac{1}{\sqrt{g_{22}}}\frac{\partial^2 g_{22}}{\partial x_1^2}\right]dx_1\,dx_2$$

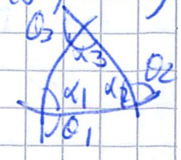
Since $\frac{\partial^2\sqrt{g_{22}}}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{1}{2}\frac{1}{\sqrt{g_{22}}}\frac{\partial g_{22}}{\partial x_1}\right)$

and $K = \frac{-1}{\sqrt{g_{22}}}\frac{\partial^2\sqrt{g_{22}}}{\partial x_1^2} = \frac{1}{4}\frac{1}{g_{22}^2}\left(\frac{\partial g_{22}}{\partial x_1}\right)^2 - \frac{1}{2}\frac{1}{g_{22}}\frac{\partial^2 g_{22}}{\partial x_1^2}$

$$d\left(\frac{\partial\sqrt{g_{22}}}{\partial x_1}\,dx_2\right) = -K\,d\sigma$$

Example If γ is a geodesic polygon ($k_g(t)\equiv 0$)

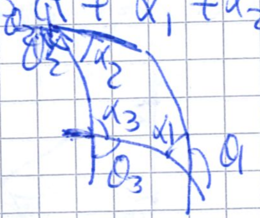
$$\iint K\,d\sigma = 2\pi - \sum\theta_i$$



Spherical triangles $K \equiv$ $A_T = 2\pi - 3\pi + \alpha_1 + \alpha_2 + \alpha_3$

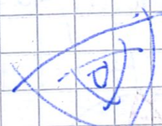
Hyperbolic $k \equiv -1$ $\quad -A = 2\pi - 3\pi + \alpha_1 + \alpha_2 + \alpha_3$

$$A = \pi - (\alpha_1 + \alpha_2 + \alpha_3)$$



Example: Gauss

Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of geodesic triangles on a surface S that converges to a pt P



There exist a neighbourhood W of P s.t. $\forall n \geq n_0$ $T_n \subseteq W$, if the triangle T_n has inner angles $\alpha_n, \beta_n, \gamma_n$ then

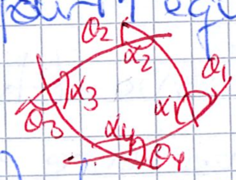
$$K(P) = \lim_{n \to \infty} \frac{\alpha_n + \beta_n + \gamma_n - \pi}{Ar(T_n)}$$

Example

1) Consider the hyperbolic plane with constant Gauss curvature $k = -1$

a) There is no geodesic quadrilateral in the hyperbolic plane with three exterior angles that are $\pi/2$ and the fourth equals $\pi/6$

b) There is geodesic quadrilateral with three interior angles that are $\pi/2$ and the fourth equals $\pi/6$



By Gauss-Bonnet Theorem

$$-A(Q) = \iint_Q K\, dA = 2\pi - (\theta_1 + \theta_2 + \theta_3 + \theta_4) = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - 2\pi$$

a) $A(Q) = 10\pi - 2\pi$ impossible

b) $A(Q) = 2\pi - \frac{10\pi}{6} = \frac{\pi}{3}$

Such a quadrilateral (all them are isometric)

Example 2 Let S be a surface with negative Gauss curvature in all its points

a) Is there a geodesic γ in S with a multiple pt that bounds a region R homeomorphic to a disc

Part a)

Part b)

b) Are there two geodesics γ_1, γ_2 that intersect in two points and enclose a region R homeomorphic to a disc?

Part a) By Gauß-Bonnet

$$0 > \iint_R K \, dA = 2\pi - \theta > 0 \quad /\text{impossible}$$

$K(x_1, x_2) < 0$

Part b) Again by Gauß-Bonnet

$$0 > \iint_R K \, dA = 2\pi - \theta_1 - \theta_2 > 0 \quad /\text{impossible}$$

$K(x_1, x_2) < 0$ $\qquad \theta_1, \theta_2 < \pi$

Observe that it is possible for positive Gauss curvature: Consider the sphere and two meridians forming a lens  as above

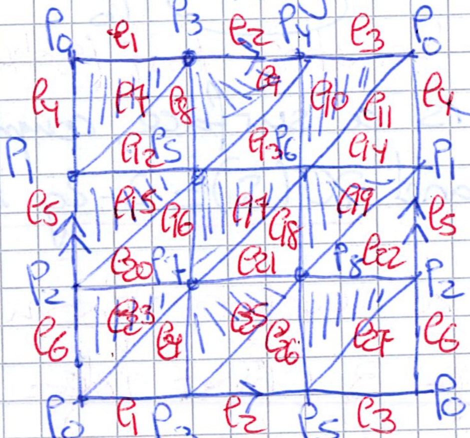
Global Theorem ~~of~~ for Gauß-Bonnet

Def Given a surface S, a triangulation of S is a family $\{T_i\}_{i \in I} = \mathcal{T}$ of (closed) triangles s.t

i) $S = \bigcup_{i \in I} T_i$

ii) Given two different triangles T_i and T_j then $T_i \cap T_j$ is empty, a vertice of a triangle or a ~~sedge~~ edge of a triangle with its end-points (vertices)

ii) ~~Each~~ Any edge of any triangle in $\mathcal{T}$ belongs to exactly two triangles



Triangulation of a torus with 18 triangles, 9 vertices and 27 edges

Def Let S be a compact surface and let $\mathcal{T} = \{T_i\}_{i=1}^{n}$ be a finite triangulation of S with n_v vertices, n_e edges and n_f triangles. The Euler characteristic $\chi_{\mathcal{T}}(S)$ of S with respect the triangulation is $\boxed{\chi_{\mathcal{T}}(S) = n_v - n_e + n_f}$

Exactly what we have learnt in the first course graph theory for planar graphs $n_v - n_e + n_f = 2$ since $\chi_{\mathcal{T}}(S^2) = 2$

Th. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two finite triangulations of a the

same compact surface S, then $\chi_{T_1}(S) = \chi_{T_2}(S)$
It is called the Euler characteristic $\chi(S)$

Th (Kato) Every compact surface admits a finite triangulation

Th If S is a compact, orientable surface, then there is a non-negative integer $g(S)$ s.t

$$\boxed{\chi(S) = 2 - 2g(S)}$$

$g(S)$ is called the genus of the surface S

For instance $g(S^2) = 0$, $\chi(S^2) = 2$

$g(T_0) = 1$ $\quad \chi(T_0) = 2 - 2 = 0$  T_0

But there is a whole plane of non-isometric tori (Schwartz)

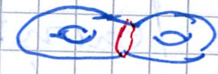
In general given $g \geq 2$ there is a surface, compact, orientable, S s.t $g(S) = g$ and $\chi(S) = 2 - 2g < 0$

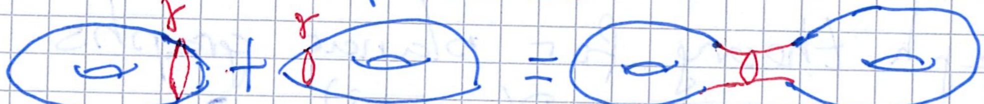
The different metrics on such a surface form a space of dimension $3g-3$ (first idea was Riemann but proved by Teichmüller)

We can construct such a surface using the connected sum of $g \geq 2$ tori (or pasting $2g-2$ pairs of pants)

Th Let S be a compact differentiable surface

$$\boxed{\iint_S k \, d\sigma = 2\pi \chi(S)}$$

Construction of 



Pair of pants