Chapter G (as in greater or equal 2)

In general (Riemann): one takes a (multi)-valued function f, and the question is: what is the most natural surface to be the domain of definition of f (meromorphic and single-valued)

We have seen that for rational functions this surface is $\hat{\mathbb{C}}$, (and almost seen) that for elliptic functions are the tori.

So <u>Def</u> Given a domain D in $\hat{\mathbb{C}}$. A <u>function element</u> is a pair (D, f) where $f: D \to \hat{\mathbb{C}}$ single-valued, meromorphic funct

From complex analysis we know that i) if (D, f) and (D, g) funct. elements s.t. $f \equiv g$ on an open $\emptyset \neq U \subset D$, then $f \equiv g$ on D.

ii) Given funct. elements (D_1, f_1) and (D_2, f_2) s.t. $D_1 \cap D_2 \neq \emptyset$ there is at most one meromorphic funct $f_2 : D_2 \to \hat{\mathbb{C}}$ s.t. $f_1 \equiv f_2$ on $D_1 \cap D_2$: the <u>direct meromorphic continuation</u>. Observe that $(D_1, f_1) \sim (D_2, f_2)$ is not transitive!!

When one can construct a sequence of direct meromorphic cont. $(D_1, f_1) \sim (D_2, f_2) \sim \dots \sim (D_n, f_n) \sim \dots$ that gives a single-valued meromorphic cont. on $\cup D_n$, one has a <u>meromorphic continuation</u>

Given a funct. element (D, f) and ∂D, whether one can continue beyond $c \in \partial D$ or not depends on f and D.

<u>Def</u> Let (D, f) be a funct. element, let $a \in D$, and let γ be a path in $\hat{\mathbb{C}}$ from a to $b \in \hat{\mathbb{C}}$. A <u>meromorphic continuation</u> of (D, f) along γ is a finite sequence of direct meromorphic cont. $(D, f) \sim (D_1, f_1) \sim \dots \sim (D_n, f_n)$ s.t

i) each domain is an open disc with $a \in D_1 \subseteq D$
ii) there is a subdivision $0 = s_0 < s_1 < \dots < s_n = 1$ of I ($\gamma: I \to \hat{\mathbb{C}}$)
s.t $\gamma([s_{i-1}, s_i]) \subseteq D_i$, $i = 1, 2, \dots$

When having meromorphic cont. along a path γ, the ~~function~~ value $f_n(b)$ depends on the initial function f and the path γ.

Th (Monodromy) If the paths γ_1 and γ_2 are homotopic then the resulting meromorphic cont. are identically equal in some neighborhood of b.

If E is a simply connected domain in $\hat{\mathbb{C}}$, given a funct element (D, f), $D \subseteq E$, if (D, f) can be meromorphically continued along all paths in E starting at some pt $a \in D$, then there is a direct meromorphic cont $(E, g) \sim (D, f)$

We have seen the definition of (abstract) R.S. we define now the morphisms:

Def. Let X be a R.S. A funct. $f: X \to \mathbb{C}$ is called analytic if for every chart (U, φ) on X, $f\varphi^{-1}: \varphi(U) \to \mathbb{C}$ analytic (and so continuous)

Observe if (V, ψ) is a different chart in a compatible atlas and $U \cap V \neq \emptyset$ then $f\psi^{-1} = f\varphi^{-1}\varphi\psi^{-1}$, also analytic

Example i) Analytic functions on $\mathbb{C} = X$
ii) Consider $\hat{\mathbb{C}}$ with $(\mathbb{C}, \varphi_N)$ and $(\hat{\mathbb{C}} \setminus \{0\}, \varphi_S)$. A funct $f: \hat{\mathbb{C}} \to \mathbb{C}$ analytic if both $f\varphi_N^{-1}(z) = \underline{f(z)}$ and $f\varphi_S^{-1}(z) = \underline{f(1/z)}$ analytic

Def Let X_1 and X_2 be R.S. A continuous function $f: X_1 \to X_2$ is holomorphic if whenever (U_1, φ_1) and (U_2, φ_2) charts for X_1 and X_2 with $U_1 \cap f^{-1}(U_2) \neq \emptyset$ the function

$$\varphi_2 f \varphi_1^{-1}: \varphi_1(U_1 \cap f^{-1}(U_2)) \longrightarrow \mathbb{C} \text{ analytic}$$

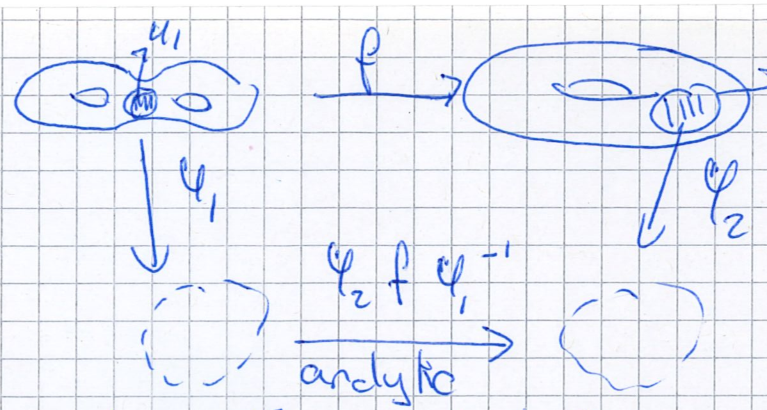
$D \cap E = F$ is a disc s.t $[f]_a = m = [g]_a$, $f \equiv g$ on F
$[f]_b = [g]_b \in A \cap B$ $\forall b \in F$ $\exists D(m) \subseteq A \cap B$.

As the function $\Psi : \mathcal{M} \longrightarrow \overline{\mathbb{C}}$ $\Psi[f]_a = a$ is continuous, $\mathcal{M}$ is Hausdorff. The restriction $\Psi_{D,m} : D(m) \longrightarrow D$ is homeomorphism, so for $\Psi(m) \neq \infty$ we have a chart at m. If $\Psi(m) = \infty$ then $J \circ \Psi_{D,m}$ is a chart (homeomorphism between $D(m)$ and an open disc of 0). Transition funct. are Id or J | analytic!!! $\mathcal{M}$ is a R.S. and Ψ meromorphic

Consider the function $\phi : \mathcal{M} \to \overline{\mathbb{C}}$ $\phi(m) = f(a)$ and having a chart $(D(m), \Psi_{D,m})$, $m = [D, f]_a$ $a \neq \infty$ f meromorphic on $D \subseteq \mathbb{C}$, and $\phi \circ \Psi_{D,m}^{-1} : D \longrightarrow \overline{\mathbb{C}}$ $(z \mapsto f(z))$ meromorphic. For $a = \infty$, $m = [D, f]_\infty$; $\phi \circ \Psi_{D,m}^{-1} \circ J : J(D) \to \overline{\mathbb{C}}$ $(z \to f(1/z))$ $0 \in J(D)$) meromorphic. $\phi : \mathcal{M} \to \overline{\mathbb{C}}$ meromorphic

Now meromorphic cont (as much as possible) = maximal ctt..

Consider $m = [f]_a \in \mathcal{M}$ and γ a path in $\overline{\mathbb{C}}$ starting at $a = \Psi(m)$. If some funct element in m can be continued along γ, then at each pt $\gamma(t)$, $t \in I$ this continuation gives a germ $\tilde{\gamma}(t)$ s.t $\Psi(\tilde{\gamma}(t)) = \gamma(t)$, the function $\tilde{\gamma} : I \xrightarrow{\text{cont}} \mathcal{M}$ is a path in $\mathcal{M}$ starting at m. If A open set in $\mathcal{M}$ and $t_0 \in \tilde{\gamma}^{-1}(A)$, $\tilde{\gamma}(t_0) = n = [f]_b$, $b = \gamma(t_0)$ and $D(n) \subseteq A$ for some D centred at b. For t near enough to $\gamma(t) \in D$ and $\tilde{\gamma}(t) = [f]_{\gamma(t)} \in D(n) \subseteq A$ so $t \in \tilde{\gamma}^{-1}(A)$
Conversely if $\tilde{\gamma}$ is a path in $\mathcal{M}$ starting at m there is a path $\Psi \circ \tilde{\gamma} = \gamma$ in $\overline{\mathbb{C}}$ giving a meromorphic continuation of $m = \tilde{\gamma}(0)$ along γ.



$\varphi \circ f : X_1 \setminus f^{-1}(a) \to \mathbb{C}$ analytic

$\psi_s \circ f : X_2 \setminus f^{-1}(0) \to \mathbb{C}$

When X_2 is $\hat{\mathbb{C}}$, we called f meromorphic

Examples i) Meromorphic functions are meromorphic

ii) $\wp : T = \mathbb{C}/\Omega \to \hat{\mathbb{C}}$ meromorphic, in general elliptic funct.

iii) $p : \mathbb{C} \to \mathbb{C}/\Omega$ holomorphic (for small enough open charts U_V $\varphi_V \circ p \circ Id^{-1} = Id : V \to V$ analytic.

Meromorphic functions $\mathcal{M}(X) = \{ f : X \to \hat{\mathbb{C}} \text{ meromorphic} \}$ form a field.

$\mathcal{M}(X) \cong \mathbb{C}(t, \varphi)$, t??, φ?? function field??

Def Given $a \in \hat{\mathbb{C}}$, let $\mathcal{F}_a = \{ f$ meromorphic on some open neighbhd of $a \}$

If $f, g \in \mathcal{F}_a$, we write $f \sim_a g$ if $f \equiv g$ on some open neighbhd of a. The equivalent class $[f]_a$ is called the germ of f at a

Now, germs represent single-valued meromorphic funct.

$\mathcal{M} = \{ [f]_a , a \in \hat{\mathbb{C}} \}$.

$\mathcal{M}$ is a R.S and the surface X of any multivalued funct f is an open subset of $\mathcal{M}$

i) Topology for $\mathcal{M}$. Let $m = [f]_a$ a germ, f meromorphic on an open disc D_a (centred at a). The D-neighboorhood of m; $D(m)$ in $\mathcal{M}$ is $D(m) = \{ [f]_b \mid b \in D \}$. Only needed to see A, B open sets $A \cap B$ also open. Let $m \in A \cap B$

$m = [D, f]_a$ D centred at a, and $m = [E, g]_a$, E centred at a

$E(m) \subseteq B$. Now $[f]_b \in A$ $\forall b \in D$, $[g]_b \in B$ $\forall b \in E$

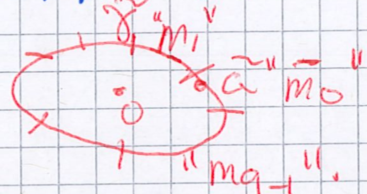
The connected component of $\mathcal{M}$, $\underline{\mathcal{M}(m)}$, formed by all the germs obtained by meromorphic cont. of m along a path γ in $\hat{\mathbb{C}}$ is the unbranched R.S of m.

If $A(z,w)$ is a single-valued function of two variables z and w, then the unbranched R.S $\mathcal{M}_A$ of $A(z,w)=0$ is the largest open subset of $\mathcal{M}$ $A(\pi(m), \phi(m))=0$ i.e the germs $m=[D,f]_a$ s.t z and $f(z)$ satisfy $A(z, f(z))=0$ $\underline{\forall z \in D}$.

Now, if $A(z,w)$ is a polynomial, whenever $f(z)$ meromorphic at $a \in \hat{\mathbb{C}}$, $g(z)=A(z, f(z))$ meromorphic at a. If $m=[f]_a \in \mathcal{M}_A$, $g\equiv 0$ near a. Continuation of m along γ in $\hat{\mathbb{C}}$ starting at a induces a continuation of g along γ, but $g\equiv 0$ near a and continuation along a path is unique, but we know that the zero-function is a continuation of g along γ. So $g(z)=A(z,w)\equiv 0$ along γ. Then $\mathcal{M}(m) \subseteq \mathcal{M}_A$.

Finally one must attach branch-points to $\mathcal{M}$. First consider branch-points at 0. Let D be a disc $|z|<\varepsilon$ and $a \in D \setminus \{0\}$. Assume that a germ $m_0=[f]_a$ can be continued analytically along all paths in $E = D\setminus\{0\}$. If γ is a loop from a to a enclosing 0 with winding number $n_0(\gamma)=1$ about 0, then by continuing m_0 along γ one obtains $m_n=[f_n]_a$ at a. Moreover if $m_n=m_0$ for some n and let q the least positive integer s.t $m_q=m_0$ then one has $,\dots, m_0, m_1, \dots, m_{q-1}, m_0, m_1, \dots$ with period q and the germs $m_0, m_1, \dots, m_{q-1}$ all distinct. We have also $\pi_1(E)=\mathbb{Z}=\langle[\gamma]\rangle$, so $m_0, \dots, m_{q-1}$

are the only germs at a that can be obtained by continuation in E. Taking $z \in E$, by continuation from a to z (along γ) there is at least q germs at z, and so exactly q germs $[g_0], \dots, [g_{q-1}]$, so one has a q-valued analytic function f ~~along~~ on E, with branches $g_0, g_1, \dots, g_{q-1}$ at z. One can represent $f(z)$ as a single-valued analytic funct $F(\zeta)$ of $\zeta = z^{1/q}$ near $\zeta = 0$. Let $\tilde{E} = \{\zeta \in \mathbb{C} \mid 0 < |\zeta^q| < \varepsilon\}$, define $\theta : \tilde{E} \to E$, $\theta(\zeta) = \zeta^q$ and choose $\tilde{a} \in \theta^{-1}(a)$. The function $F_0(\zeta) = f_0 \circ \theta(\zeta)$ analytic near $\tilde{a}$, with germ $\tilde{m}_0 = [F_0]_{\tilde{a}}$ at $\tilde{a}$. One can continue $\tilde{m}_0$ analytically along all paths in $\tilde{E}$ obtaining $F(\zeta) = (f \circ \theta)(\zeta) = f(\zeta^q)$ on $\tilde{E}$, and F is single-valued since $\pi_1(\tilde{E}) = \cancel{\mathbb{Z}} = \langle \tilde{\gamma} \rangle$

(the picture 

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$$F(\zeta) = \sum_{-\infty}^{\infty} c_r \zeta^r \text{ on } \tilde{E} \qquad f(z) = \sum c_r z^{r/q} \text{ on } E$$

If there exist N s.t. $c_N \neq 0$ and $c_r = 0$ $\forall r < N$, then F is meromorphic at 0, $F(0) = c_0$ or ∞ ($N \geq 0$ or $N < 0$). If $q = 1$ then $f = F$ single-valued and meromorphic at 0, so $[f]_0$ belongs to $\mathcal{U}$. Assume that $q > 1$ and attach $[f]_0$ at $\mathcal{U}$. $[f]_0$ is a branch-point of order $q-1$ (for $\infty \in \bar{\mathbb{C}}$, take $\zeta^q = 1/z$). Let X be now $\mathcal{U}$ and all the attached pts in this way and extend $\pi, \phi : X \to \bar{\mathbb{C}}$ by $\pi([f]_c) = c$ and $\phi([f]_c) = F(0)$. If D open disc centred at c the D-neighbhd of $[f]_c$, $D([f]_c)$ consists of $[f]_c$ and $\forall a \in D \setminus \{0\}$ the germs $m_0, \dots, m_{q-1}$.
$\mathcal{U}$ is an open subset of X and $X \setminus \mathcal{U}$ is discrete

Also X is a R.S using ζ as local coordinates near branch points $[f]_c$. For $c=0$ $\zeta \in \mathring{E}$ is the unique germ $[g_j]_z$ at $z=\zeta^q$ s.t $g_j \circ \theta = F$ near ζ, and $[f]_c$ has coordinate ζ, thus each root ζ of $\zeta^q = z$ corresponds to the germ $[g_j]_z$ at z. The map $[g_j]_z \to \zeta$ is a homeomorphism between $D([f]_c)$ and $\mathring{D} = \mathring{E} \cup \{0\}$. Observe that the only charts that overlap this belong to $\mathcal{M}$ and the transition functions $\zeta \to z = \zeta^q$ and $z \mapsto \zeta = z^{1/q}$ analytic since $z \neq 0$. Similarly for $0 \neq c \in \mathbb{C}$, and $c = \infty$)
Each branch point is adjoined to a unique connected component $\mathcal{M}(m)$ of $\mathcal{M}$, where $m \in D([f]_c)$. There is a bijection between connected components of X and of $\mathcal{M}$. For a polynomial $A(z,w)=0$ X_A is the largest open subset of X on which $A(\pi, \phi) = 0$. Then
$\mathcal{M}_A = \mathcal{M} \cap X_A$.

One has ($A = a_0(z)w^n + \dots + a_{n-1}(z)w + a_n(z)$)

I) If $A = \prod A_i^{n_i}$, A_i irreducible, then $X_A = \bigcup X_{A_i}$

II) If $a \in \hat{\mathbb{C}} \setminus \{\text{critical pts of } A\}$, then there exists D centred at a and analytic function elements (D, f_i) $1 \leq i \leq n$ s.t $f_i(a) = w_i$ $i = 1, \dots, n$ and for each $z \in D$ the solutions of $A(z,w)=0$ are $w = f_i(z)$ all simple and distinct

III) If E simply connected in $\hat{\mathbb{C}} \setminus \{\text{critical pts of } A\}$, there are single-valued analytic functions $f_1, \dots, f_n$ on E s.t the solutions of $A(z,w)=0$ are $w = f_i(z)$, all simple and distinct

IV) If $A(z,w)$ irreducible, then X_A connected

V) X_A compact

VI) (Very difficult) Any ~~abstract~~ compact RS is the R.S of some algebraic function $A(z,w)=0$.

Ya hemos visto (en otros minicursos) que si X is the RS X_A of an irreducible alg. equation $A(z,w)=0$ of deg n in w, and if the branch-pts have orders $n_1, \dots, n_r$, then the genus of X is given by $\boxed{g=1-n+\frac{1}{2}\sum n_i}$ or $\boxed{2-2g=n\left(2+\sum\frac{1-q_i}{n}\right)}$ where q_i the mult. of the branch points.

The Riemann-Hurwitz formula, since one uses that the function $t: X \to \hat{\mathbb{C}}$ is a branched covering

Examples: i) $A(z,w)=w^q-z$; $n=q$ and there are two branch pts: 0 and ∞ of orders $q-1$ and $q-1$ so
$$2-2g=q\left(2-1-1+\frac{2}{q}\right) \;;\; g=0 \text{ as we knew.}$$

ii) $A(z,w)=w^2-(z-a_1)\cdots(z-a_m)$, m odd, $n=2$ and the branch-points are $a_1, \dots, a_m, \infty$ of order 1
so $g=-1+\frac{m+1}{2}$; $\boxed{g=\frac{m-1}{2}}$

Observation. Let $f: X_1 \to X_2$ be a holomorphic homeomorphism (as if $F: U \to V$ analytic homeomorphism between open subsets of $\mathbb{C}$, implies that $F^{-1}: V \to U$ also analytic) we have that $f^{-1}: X_2 \to X_1$ is also holomorphic homeomorphism. In such situation we say that X_1 and X_2 are conformally equivalent, $X_1 \simeq X_2$.

Example i) $\mathbb{D}=\{z\in\mathbb{C} \mid |z|<1\}$ and the upper-half plane $\mathbb{H}=\{z\in\mathbb{C} \mid \operatorname{Im} z>0\}$ are conformally equivalent under the homeomorphism $T: \mathbb{H} \to \mathbb{D}$; $T(z)=\frac{z-i}{z+i}$

ii) $\mathbb{C}$ and $\mathbb{D}$ are homeomorphic but no conformally equiv.

The only simply connected open subsets of $\mathbb{C}$ are $\mathbb{C}$ or a set conformally equiv. to $\mathbb{D}$.

We can see in Scott's paper:

Th (Uniformisation) Every simply connected RS is conformally equivalent to

i) $\hat{\mathbb{C}}$, ii) $\mathbb{C}$ or iii) $\mathbb{H}$ (or $\mathbb{D}$)

As a consequence a RS homeomorphic to $\hat{\mathbb{C}}$ must be conformally equiv to $\hat{\mathbb{C}}$. The only structure of RS on a top. sphere is $\hat{\mathbb{C}}$. (projective line)

We have learnt before that $Aut(\hat{\mathbb{C}}) = PSL(2,\mathbb{C})$ and $Aut(\mathbb{C}) = \{z \mapsto az+b \mid a,b \in \mathbb{C}, a \neq 0\}$ and

Exercise $Aut(\mathbb{H}) = PSL(2,\mathbb{R})$

We put our attention on tori $\mathbb{C}/\Omega_1$, $\mathbb{C}/\Omega_2$.
Two lattices Ω and Ω' are similar iff $\exists a \neq 0$ s.t. $a\Omega = \Omega'$.

Th The holomorphic functions $f: \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ are the transf. of the form $f_{a,b}[z] = [az+b]'$, with $a,b \in \mathbb{C}$ $a \neq 0$, $a\Omega \subseteq \Omega'$. $\mathbb{C}/\Omega$ and $\mathbb{C}/\Omega'$ are conformally equiv iff Ω and Ω' similar (i.e $\exists \gamma \in PSL(2,\mathbb{Z})$ taking Ω to Ω')

Proof We will see that if $f: \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ holomorphic, there is an automorphism $\tilde{f}$ of $\mathbb{C}$ s.t. $f \circ p = p' \circ \tilde{f}$, with p and p' the natural projections.
Consider the atlases $\mathcal{V}$ and $\mathcal{V}'$ of $\mathbb{C}$ given by the open discs of radius $d/2$ and $d'/2$ resp.

If $f[z]=[z']'$ we have $V\in\mathcal{V}$ of z and $V'\in\mathcal{V}'$ of z'

$$\begin{array}{ccc} V & \longrightarrow & V' \\ p \downarrow & & \downarrow p' \\ U & \xrightarrow{f} & U' \end{array}$$

$p|_V$ and $p'|_{V'}$ are homeomorphisms onto U and U' resp (the inverse of the charts we defined before)

$F = p'^{-1} f p : p^{-1}(U \cap f^{-1}(U')) \longrightarrow p'^{-1}(U' \cap f(U))$ analytic

At each $z\in\mathbb{C}$ we have a set of analytic germs $[F+\omega']_z$

As $\mathbb{C}$ is simply connected, any of these germs extends to a single-valued analytic function $\tilde{f}:\mathbb{C}\to\mathbb{C}$, the <u>lift of f</u>. Since $f\circ p = p'\circ\tilde{f}$; $f[z]=[\tilde{f}(z)]'$ $\forall z\in\mathbb{C}$ and for any fixed $\omega\in\Omega$ $\tilde{f}(z+\omega)=\tilde{f}(z)+\omega'_z$, $\omega'_z\in\Omega'$
$z\mapsto\omega'_z$ is continuous and as Ω' discrete, constant so $\tilde{f}(z+\omega)=\tilde{f}(z)+\omega'$; $d\tilde{f}/dz$ is elliptic for Ω and analytic so constant so $\tilde{f}(z)=az+b$ so $f(z)=f_{a,b}(z)$. $\forall\omega\in\Omega$ $f[z+\omega]=f[z]$ so $[az+a\omega+b]' = [az+b]'$, thus $a\omega\in\Omega'$, $a\Omega\subseteq\Omega'$. If $a\Omega\subseteq\Omega'$ then $f_{a,b}$ is holomorphic (as Exercise)
If $f_{a,b}$ conformal homeomorphism $[z]'\mapsto[a^{-1}(z-b)]$ with $a^{-1}\Omega'\subseteq\Omega$ so $\Omega'\subseteq a\Omega$.

And $\mathrm{Aut}(\mathbb{C}/\Omega)=\{f_{a,b}:[z]\mapsto[az+b] \mid a,b\in\mathbb{C},\ a\neq 0,\ a\Omega=\Omega\}$

<u>Exercise</u> $\mathrm{Aut}(\mathbb{C}/\Omega)$ has a normal subgroup isomorphic to $\mathbb{C}/\Omega$

of course. Let $p(z)=4z^3-g_2z-g_3$, where $g_2=60\sum'\omega^{-4}$ and $g_3=140\sum'\omega^{-6}$, and let X be the RS of $\sqrt{p(z)}$. Then $X\simeq\mathbb{C}/\Omega$

$\alpha:\mathbb{C}/\Omega\longrightarrow X$
$t\longmapsto[f]_c\in X$ $c=\wp(t)$, f local branch of $\sqrt{p}$
$f(c)=\wp'(t)$.

Th (Structure) If X is a RS and $\tilde{X}$ is a covering of S, then there is a unique structure on $\tilde{X}$ s.t $p: \tilde{X} \to X$ holomorphic. Moreover each covering transf of $(\tilde{X}, p)$ is an automorphism of $\tilde{X}$

Proof. Take elementary neighborhoods V in $\tilde{X}$ small enough that each is mapped by p homeomorphically onto $U \subseteq X$, (U, φ) chart on X. An atlas for $\tilde{X}$ has charts $(V, \varphi \circ p)$ (call they). Given two such charts $(V_i, \varphi_i \circ p)$ and $(V_j, \varphi_j \circ p)$ with $V_i \cap V_j \neq \emptyset$ $(\varphi_j \circ p) \circ (\varphi_i \circ p)^{-1} = \varphi_j \circ \varphi_i^{-1}$ analytic, and p holomorphic. Also any ~~atlas~~ compatible atlas on $\tilde{X}$ making p holomorphic is compatible with the above one (using just the definition)

If g is a covering transformation, taking charts $(V_{\tilde{x}}, \varphi_{p(\tilde{x})} \circ p)$ and $(V_{g(\tilde{x})}, \varphi_{pg(\tilde{x})} \circ p)$ of $\tilde{x}$ and $g(\tilde{x})$ one has

$$(\varphi_{pg(\tilde{x})} \circ p) \circ g \circ (\varphi_{p(\tilde{x})} \circ p)^{-1} = \varphi_{pg(\tilde{x})} \circ \underbrace{p \circ g}_{p} \circ p^{-1} \circ \varphi_{p(\tilde{x})}^{-1} = \underbrace{\varphi_{pg(\tilde{x})} \circ \varphi_{p(\tilde{x})}^{-1}}_{\text{analytic}}$$

Th If X is a connected RS not conformally equivalent to $\hat{\mathbb{C}}$, $\mathbb{C}$, a torus $\mathbb{C}/\Omega$ or $\mathbb{C} \setminus \{0\}$, then the universal covering of X is $\mathbb{H}$. Moreover $X \simeq \mathbb{H}/G$ for some subgroup G of $PSL(2, \mathbb{R})$ acting freely discontinuously on $\mathbb{H}$.

G is isomorphic to the fundamental group of a surface

Proof By the theorem above the universal covering $\mathcal{U}$ of X has a unique complex structure, and it is regular. So $q: X \longrightarrow \mathcal{U}/G$ ($\mathcal{U} = \hat{\mathbb{C}}, \mathbb{C}, \mathbb{H}$) and $G \leq \mathrm{Aut}(\mathcal{U})$ and we transfer the complex structure of X (given by $\{(U, \varphi)\}$) to $\mathcal{U}/G$ $\{(q(U), \varphi p^{-1})\}$ analytic atlas and $X \simeq \mathcal{U}/G$

Now, any non-identity automorphism of $\hat{\mathbb{C}}$ has at least a fixed point, so $G = \{Id\}$ and $X \simeq \hat{\mathbb{C}}$

If $\mathcal{U} = \mathbb{C}$, $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az+b\}$, and G is a group of translations, and G has to be discrete, i.e. trivial, an infinite cyclic group or a lattice. Then $X \cong \mathbb{C}$, $X \cong \mathbb{C} \setminus \{0\}$ or $X \cong$ torus. All excluded.
So $\mathcal{U} = \mathbb{H}$ and $G \leq PSL(2, \mathbb{R})$ discrete.

Exercise : 1