element a of G to create a permutation t_a of G (as a set) in such a way that G itself is isomorphic to the group of these permutations. The proof of Theorem 14.5.6 is similar. □

The proof of Theorem 14.5.6 Let G be any group. For each g in G let $\theta_g : G \to G$ be the map defined by $\theta_g(h) = gh$ (informally, θ_g is the instruction 'multiply on the left by g'). We begin by showing that each θ_g is a permutation of G. Certainly, θ_g is a map of G into itself. Next, if $\theta_g(f) = \theta_g(h)$ then $gf = gh$ so that $f = h$; thus θ_g is injective. Finally, take any h in G and notice that $\theta_g(g^{-1}h) = gg^{-1}h = h$ so that θ_g is surjective. Thus each θ_g is a permutation of G.

It follows that we have just constructed a map θ that takes g to θ_g, and this is a map from G to the group, say $\mathcal{P}$, of permutations of G. Now it is easy to see that θ is a homomorphism from G to $\mathcal{P}$. Indeed, given g and h in G,

$$\theta_{gh}(f) = (gh)f = g(hf) = \theta_g(hf) = \theta_g\theta_h(f),$$

and as this holds for all f, we see that $\theta_{gh} = \theta_g\theta_h$. Thus $\theta : G \to \mathcal{P}$ is a homomorphism. Finally, let Γ be the image of G under θ. As G is a group and θ is a homomorphism, we see that Γ is a subgroup of $\mathcal{P}$. By definition, θ maps G onto Γ; thus θ is an isomorphism from G onto the subgroup Γ of $\mathcal{P}$. □

Exercise 14.5

1. Let G be the group of transformations $\{I, f, g, h\}$, where $f(z) = -z$, $g(z) = \bar{z}$ and $h(z) = -\bar{z}$. Show that the map $v \mapsto fv$ is a permutation of G.
2. Let G be any group, and let G act on itself as described in the proof of Cayley's theorem. Show that G acts faithfully on itself.
3. Let G be a group and H a subgroup of G. Show that G acts on the set of left cosets by the rule that g (in G) takes hH to ghH; equivalently, $g(hH) = ghH$. Now H itself is a left coset ($= eH$), so we can ask for the subgroup of elements of G that fix H. Show that this subgroup is H; thus any subgroup of any group arises as the stabilizer of some group action.

15

Hyperbolic geometry

15.1 The hyperbolic plane

In the earlier chapters we have discussed both Euclidean geometry and spherical (non-Euclidean) geometry, and in this last chapter we discuss a second type of non-Euclidean geometry, namely hyperbolic geometry. Gauss introduced the term *non-Euclidean geometry* to describe a geometry which does *not* satisfy Euclid's axiom of parallels, namely that if a point P is not on a line L, then there is exactly one line through P that does not meet L. In spherical geometry, the 'lines' are the great circles, and in this case any two lines meet. Hyperbolic geometry is a geometry in which there are infinitely many lines through the point P that do not meet the line L, and it was developed independently by Gauss (in Germany), Bolyai (in Hungary) and Lobatschewsky (in Russia) around 1820.

We begin by describing the points and lines of hyperbolic geometry without any reference to distance. We shall take the hyperbolic plane to be the upper half-plane $\mathcal{H} = \{x + iy : y > 0\}$ in $\mathbb{C}$. Notice that the real axis $\mathbb{R}$ is not part of $\mathcal{H}$. A *hyperbolic line* (that is, a line in the hyperbolic geometry) is a semicircle in $\mathcal{H}$ whose centre lies on $\mathbb{R}$; such semi-circles are orthogonal to $\mathbb{R}$. However, as our concept of circles includes 'straight lines' (see Chapter 14), we must also regard those straight lines that are orthogonal to $\mathbb{R}$ as hyperbolic lines. Figure 15.1.1 illustrates the hyperbolic lines in $\mathcal{H}$, and we remark that the two 'different' types of line are only different because we are viewing them from a Euclidean perspective.

We notice immediately that Euclid's Parallel Axiom fails; indeed, the two semi-circles have a common point P that does not lie on the line L; moreover, it is easy to see that there are infinitely many hyperbolic lines through P and not meeting L. It is clear, however, that *any two hyperbolic lines meet in at most one point*, and that *there is is a unique hyperbolic line through any two distinct points in $\mathcal{H}$.*

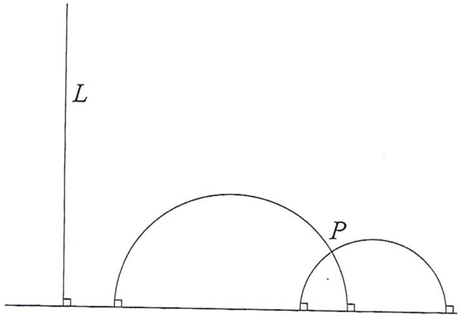


Figure 15.1.1

Now let

$$\Gamma = \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R},\ ad - bc > 0 \right\}.$$

First, we leave the reader to check that Γ is a group. Next, we note that if g is in Γ then it maps $\mathcal{H}$ into itself. Indeed, if we write $g(z) = (az+b)/(cz+d)$, and $z = x + iy$, then

$$\begin{aligned} \operatorname{Im}[g(x+iy)] &= \frac{(az+b)\overline{(cz+d)}}{(cz+d)\overline{(cz+d)}} \\ &= \frac{(ad-bc)\,y}{|cz+d|^2} > 0. \end{aligned} \tag{15.1.1}$$

Exactly the same reason shows that g also maps the lower half-plane (given by $y < 0$) into itself; thus g must also map the circle $\mathbb{R} \cup \{\infty\}$ into itself. As Γ is a group, the same holds for g^{-1}; thus

$$g(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\} = g^{-1}(\mathbb{R} \cup \{\infty\}).$$

This implies that the coefficients a, b, c and d in g may be chosen to be real (note that we cannot assert that they *are* real, for they are only determined to within a complex scalar multiple). The case $c = 0$ is easy, so we may assume that $c \neq 0$. Then, by scaling the coefficients by the factor $1/c$, we can choose these coefficients so that, in effect, $c = 1$. Then, as $g(\infty) = a$, and $g^{-1}(\infty) = -d$, we see that a and d are real. Finally, if $a = 0$ then $-b = ad - bc > 0$ so that b is real. If, however, $a \neq 0$, then $-b/a = g^{-1}(0)$ so that b is again real. To summarize: if $g(z) = (az+b)/(cz+d)$, and $g \in \Gamma$, then we may assume that a, b, c and d are real. This implies that, for each z,

2.1

$$\overline{g(z)} = \overline{\left(\frac{az+b}{cz+d}\right)} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} = \frac{a\bar{z}+b}{c\bar{z}+d} = g(\bar{z}). \tag{15.1.2}$$

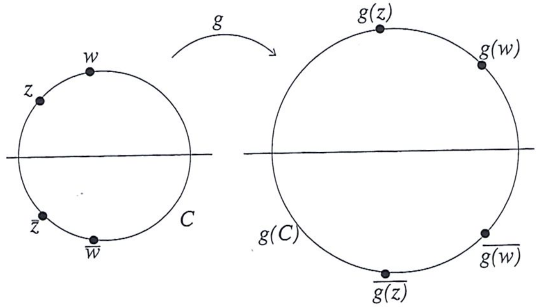


Figure 15.1.2

Let us now see how g in Γ acts on hyperbolic lines. Let z and w be distinct points of $\mathcal{H}$, and let us consider the hyperbolic line L through z and w. Now L is part of the (unique) Euclidean circle C that passes through z, w, $\bar{z}$ and $\bar{w}$, so that $g(C)$ passes through $g(z)$, $g(w)$, $g(\bar{z})$ and $g(\bar{w})$. However, (15.2.1) now implies that $g(C)$ passes through $g(z)$, $\overline{g(z)}$, $g(w)$ and $\overline{g(w)}$, so that $g(C)$ is orthogonal to $\mathbb{R}$ (see Figure 15.1.2). It follows that $g(L) = \mathcal{H} \cap g(C)$, and this proves the following result.

Theorem 15.1.1 *If $g \in \Gamma$ and L is a hyperbolic line, then $g(L)$ is a hyperbolic line.*

Theorem 15.1.1 suggests that the elements of Γ might be regarded as the rigid motions of hyperbolic geometry. This suggestion is strengthened by the fact (which will not be proved here) that any bijective map of $\mathbb{C}_\infty$ onto itself that maps circles to circles is a Möbius map of z or of $\bar{z}$; this is a type of converse of Theorem 13.3.2. Further, any Möbius map that preserves $\mathcal{H}$ must be in Γ (Exercise 15.1.1). In the next section we shall introduce a distance in $\mathcal{H}$, and we shall then see that the elements of Γ are indeed the isometries of $\mathcal{H}$.

There is a second model of the hyperbolic plane which is useful, and often preferable to the model $\mathcal{H}$. In this model the hyperbolic plane is the unit disc $\mathbb{D}$, namely $\{z : |z| < 1\}$ (see Figure 15.1.3). The Möbius map $g(z) = (z-i)/(z+i)$ maps $\mathcal{H}$ onto $\mathbb{D}$ (because $\mathcal{H}$ is given by $|z-i| < |z+i|$), and so we may take the hyperbolic lines in the model $\mathbb{D}$ to be the images under g of the hyperbolic lines in $\mathcal{H}$. Thus the hyperbolic lines in $\mathbb{D}$ are the arcs of circles in $\mathbb{D}$ whose endpoints lie on the cirle $|z| = 1$ and which are orthogonal to this circle at their endpoints. The two models $\mathcal{H}$ and $\mathbb{D}$ may be used interchangeably, and any result about one may be transferred to the other by any Möbius map that maps $\mathcal{H}$ to $\mathbb{D}$, or $\mathbb{D}$ to $\mathcal{H}$.

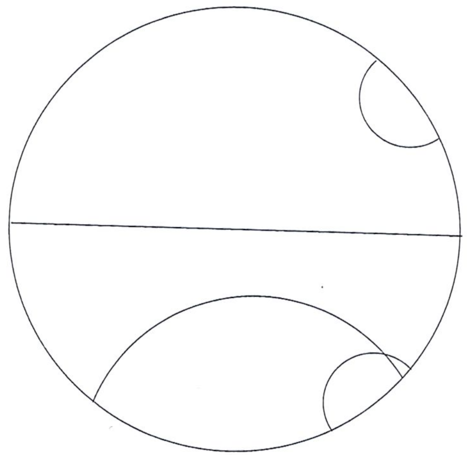


Figure 15.1.3

Exercise 15.1

1. Show that if f is a Möbius map which maps $\mathbb{R} \cup \{\infty\}$ onto itself, then f can be written in the form $f(z) = (az+b)/(cz+d)$, where a, b, c, d are real and $ad - bc \neq 0$. Show further that $f(\mathcal{H}) = \mathcal{H}$ if and only if $ad - bc > 0$.
2. Let z_1 and z_2 be distinct points in the hyperbolic plane. Show that there is a unique hyperbolic line that passes through z_1 and z_2.
3. Suppose that w_1 and w_2 are in $\mathcal{H}$. Show that there is some g in Γ such that $g(w_1) = w_2$. This shows that *the stabilizer of any point in $\mathcal{H}$ is conjugate to the stabilizer of any other point in $\mathcal{H}$.*
4. Verify the steps in the following argument. Let $g(z) = (z-i)/(z+i)$; then g maps $\mathcal{H}$ onto $\mathbb{D}$, where $\mathbb{D} = \{z : |z| < 1\}$. Note that $g(i) = 0$ and $g(-i) = \infty$. Now suppose that f maps $\mathcal{H}$ onto itself and fixes i; then f also fixes $-i$. It follows that gfg^{-1} maps $\mathbb{D}$ onto itself, and fixes 0 and ∞. Thus gfg^{-1} is a Euclidean rotation about the origin, and hence *the group of hyperbolic isometries that fix a given point w is isomorphic to the group of Euclidean rotations that fix the origin.*

15.2 The hyperbolic distance

We shall now introduce a distance in $\mathcal{H}$, and then show that the elements of Γ are isometries for this distance (in fact, they are the only orientation-preserving isometries). There are two ways to define this *hyperbolic distance*, and we shall

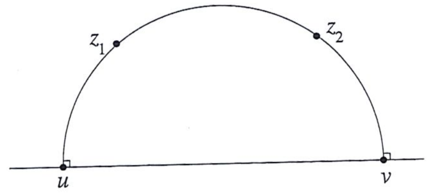


Figure 15.2.1

start with the more elementary way. Consider distinct points z_1 and z_2 in $\mathcal{H}$, and let L be the hyperbolic line through these points. Then L has endpoints u and v, say, chosen so that u, z_1, z_2 and v occur in this order along L (see Figure 15.2.1). We can find a Möbius map g in Γ such that $g(u) = 0$ and $g(v) = \infty$ (see Exercise 15.2.1); then $g(z_1) = ia$ and $g(z_2) = ib$, say, where $0 < a < b$. If we now recall that cross-ratios are invariant under Möbius maps, we have

$$[u, z_1, z_2, v] = [0, ia, ib, \infty] = b/a > 1.$$

This allows us to make the following definition.

Definition 15.2.1 The *hyperbolic distance* between z_1 and z_2 in $\mathcal{H}$ is $\log [u, z_1, z_2, v]$ when $z_1 \neq z_2$, and zero otherwise. We denote this distance by $\rho(z_1, z_2)$. □

Theorem 15.2.2 *The elements of Γ preserve the hyperbolic distance between two points in $\mathcal{H}$.*

Proof This is immediate because the hyperbolic distance is defined as a cross-ratio, cross-ratios are invariant under Möbius maps, and each g in Γ is a Möbius map. □

Notice that if, in Definition 15.2.1, we have $z_1 = ia$ and $z_2 = ib$, where $0 < a < b$, then

$$\rho(ia, ib) = \log[0, ia, ib, \infty] = \log b/a$$

(see Definition 13.4.1). This leads to the following more general result.

Theorem 15.2.3 *The hyperbolic distance is additive along hyperbolic lines.*

Proof Suppose that z_1, z_2, z_3 lie on a hyperbolic line L with end-points u and v such that u, z_1, z_2, z_3, v occur in this order along L. We can find some g in Γ such that $g(u) = 0$ and $g(v) = \infty$, and then $g(z_j) = ia_j$, where $0 < a_1 < a_2 < a_3$.

As $\rho(z_i, z_j) = \log a_j/a_i$ when $i < j$ we see that

$$\begin{aligned}\rho(z_1, z_3) &= \log a_3/a_1 \\ &= \log a_3/a_2 + \log a_2/a_1 \\ &= \rho(z_1, z_2) + \rho(z_2, z_3).\end{aligned}$$

□

We are now in a position to give explicit formulae for the hyperbolic distance (see Exercise 15.2.2).

Theorem 15.2.4 *For z and w in $\mathcal{H}$,*

$$\sinh^2 \frac{1}{2}\rho(z, w) = \frac{|z-w|^2}{4\,\mathrm{Im}[z]\,\mathrm{Im}[w]}, \tag{15.2.1}$$

$$\cosh^2 \frac{1}{2}\rho(z, w) = \frac{|z-\bar{w}|^2}{4\,\mathrm{Im}[z]\,\mathrm{Im}[w]}. \tag{15.2.2}$$

Proof First, choose g in Γ (as above) so that $g(z) = ia$ and $g(w) = ib$, where $0 < a < b$. By applying the map $z \mapsto z/a$ (which is in Γ), we may assume that $a = 1$. Then $\rho(z, w) = \rho(i, ib) = \log b$, so that

$$\sinh^2 \frac{1}{2}\rho(z, w) = \sinh^2\left(\log\sqrt{b}\right) = \frac{(b-1)^2}{4b}. \tag{15.2.3}$$

Next, let

$$F(z, w) = \frac{|z-w|^2}{4\,\mathrm{Im}[z]\,\mathrm{Im}[w]}.$$

Then, from (13.1.2) and (15.1.1), we see that F is invariant under any g in Γ; that is,

$$F\big(g(z), g(w)\big) = F(z, w).$$

Thus

$$F(z, w) = F(i, ib) = \frac{(b-1)^2}{4b}, \tag{15.2.4}$$

and this together with (15.2.3) gives (15.2.1). The second formula (15.2.2) follows from the fact that, for all z, $\cosh^2 z = 1 + \sinh^2 z$. □

We remark that, as Theorem 15.2.4 suggests, in calculations involving the hyperbolic distance it is almost always advantageous to use the functions sinh or cosh of $\rho(z, w)$ or $\frac{1}{2}\rho(z, w)$; *only rarely is $\rho(z, w)$ used by itself.*

We end with a brief discussion of an alternative (but equivalent) way to define distance. First, we define the *hyperbolic length* of a curve γ in $\mathcal{H}$ to be the line integral

$$\int_\gamma \frac{|dz|}{y},$$

where, as usual $z = x + iy$. Now let L be the hyperbolic line through two points z and w in $\mathcal{H}$, and let σ be the arc of L that lies between z and w. It can be shown that σ has hyperbolic length $\rho(z, w)$ and, moreover, that any other curve joining z to w has a greater hyperbolic length than σ. Thus the hyperbolic line through two points does indeed give the shortest path between these points.

Finally, if g is in Γ, and $g(z) = (az+b)/(cz+d)$, then, from (13.1.2), we see that

$$|g'(z)| = \frac{|ad-bc|}{|cz+d|^2},$$

where $g'(z)$is the usual derivative of g. In conjunction with (15.1.1) this gives

$$\frac{|g'(z)|}{\mathrm{Im}[g(z)]} = \frac{1}{\mathrm{Im}[z]}.$$

This (together with the formula for a change of variable in a line integral) shows that for each g in Γ, and each curve γ, $g(\gamma)$ has the same hyperbolic length as γ.

Exercise 15.2

1. Show that for any u and v in $\mathbb{R} \cup \{\infty\}$ with $u \neq v$, there is a g in Γ with $g(u) = 0$ and $g(v) = \infty$. [Hint: apply $z \mapsto -1/(z-v)$ and then a translation.]
2. The functions sinh and cosh are defined by $\sinh z = (e^z - e^{-z})/2$ and $\cosh z = (e^z + e^{-z})/2$. Show that (a) $\cosh^2 z - \sinh^2 z = 1$, and (b) $\cosh 2z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$.
3. Find the hyperbolic distance between the points $1 + iy$ and $-1 + iy$ as a function of y. Show that for a given positive t there is a value of y such that this distance is t.
4. Let L be the Euclidean line given by $\mathrm{Im}[z] = 2$. Show that $2i$ is the point on L that is closest (as measured by the hyperbolic distance) to the point i.

15.3 Hyperbolic circles

Suppose that $w \in \mathcal{H}$, and $r > 0$. The *hyperbolic circle* with *hyperbolic centre* w and *hyperbolic radius* r is the set $\{z \in \mathcal{H} : \rho(z, w) = r\}$.

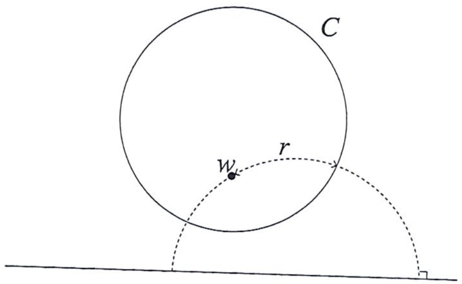


Figure 15.3.1

Theorem 15.3.1 *Each hyperbolic circle is a Euclidean circle in $\mathcal{H}$.*

Proof Let C be the hyperbolic circle with centre w and radius r. There is a map g in Γ with $g(w) = i$ (see Exercise 15.1.2), so that $g(C)$ is the hyperbolic circle with centre i and radius r. Now by Theorem 15.2.4, $z \in g(C)$ if and only if $|z - i|^2/4y = \sinh^2 \frac{1}{2}r$, where $z = x + iy$. This equation simplifies to give $x^2 + (y - \cosh r)^2 = \sinh^2 r$, so that $g(C)$ is a Euclidean circle in $\mathcal{H}$. As g^{-1} maps circles to circles, and $\mathcal{H}$ to itself, we see that $g^{-1}(g(C))$, namely C, is a Euclidean circle in $\mathcal{H}$. □

Notice that the hyperbolic centre of a hyperbolic circle is *not* the same as its Euclidean centre (and similarly for the radii); indeed, the hyperbolic circle $g(C)$ in the proof of Theorem 15.3.1 has hyperbolic centre i, and Euclidean centre $i \cosh r$ (and $\cosh r > 1$). A hyperbolic circle with centre w and hyperbolic radius r is illustrated in Figure 15.3.1. Finally, it can be shown that the length of a hyperbolic circle of hyperbolic radius r is $2\pi \sinh r$, and that its hyperbolic area (which we have not defined) is $4\pi \sinh^2(\frac{1}{2}r)$. Notice that the hyperbolic radius of a hyperbolic circle of radius r grows roughly like πe^r; in the Euclidean case, it is $2\pi r$. Finally, we mention (but do not prove) the hyperbolic counterpart of the fact that the area of a spherical triangle is π less than its angle sum.

Theorem 15.3.2 *The area of a hyperbolic triangle with angles α, β and γ is $\pi - (\alpha + \beta + \gamma)$. In particular, this area cannot exceed π.*

Exercise 15.3

1. Find the equation of the hyperbolic circle with centre $2i$ and radius e^2. Suppose that this circle meets the imaginary axis at ia and ib, where $0 < a < 2 < b$. Find a and b, and verify directly that $\rho(ia, 2i) = e^2 = \rho(2i, ib)$.

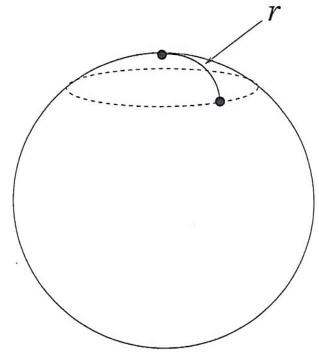


Figure 15.3.2

2. Consider the unit sphere in $\mathbb{R}^3$ as a model of spherical geometry in which distances are measured on the surface of the sphere. What is the circumference of the circle whose centre is at the 'north pole' (see Figure 15.3.2) and whose radius is r? Now compare the circumference of a circle of radius r in Euclidean geometry, spherical geometry and hyperbolic geometry.

15.4 Hyperbolic trigonometry

A hyperbolic triangle is a triangle whose sides are arcs of hyperbolic lines. We begin with the hyperbolic version of Pythagoras' theorem (see Figure 15.4.1).

Theorem 15.4.1 *Suppose that a hyperbolic triangle has sides of hyperbolic lengths a, b and c, and that the two sides of lengths a and b are orthogonal. Then $\cosh c = \cosh a \cosh b$.*

In our proof of this result we shall need to use the fact that a Möbius map is *conformal*; that is, it preserves the angles between circles. In particular, this implies that if two circles C and C' are orthogonal, and if g is any Möbius map, then $g(C)$ and $g(C')$ are orthogonal. We shall not give a proof of this (although the proof is not difficult).

Proof Let the vertices of the hyperbolic triangle be v_a, v_b and v_c, where v_a is opposite the side of length a, and so on. There is some g in Γ such that $v_c = i$ and $v_b = ik$, where $k > 1$. As g preserves the orthogonality of circles we see that g maps v_a to some point $s + it$, where $s^2 + t^2 = 1$ (see Figure 15.4.2). As g preserves hyperbolic distances, this means that we may assume that $v_a = s + it$,

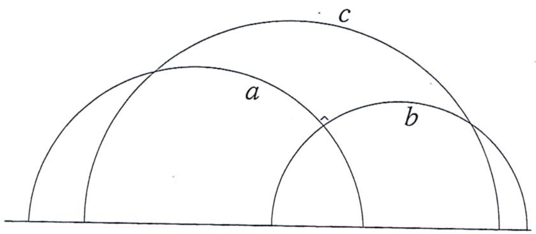


Figure 15.4.1

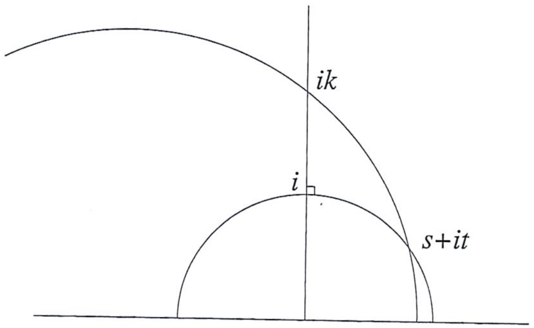


Figure 15.4.2

$v_b = ik$ and $v_c = i$. Now $\rho(i, ik) = a$, $\rho(i, s+it) = b$ and $\rho(ik, s+it) = c$. As

$$\cosh a = \frac{k^2+1}{2k}, \quad \cosh b = \frac{1}{t}, \quad \cosh c = \frac{k^2+1}{2tk},$$

the given formula follows. □

It is interesting to examine Pythagoras' theorem for small triangles, and for large triangles. As $\cosh z = 1 + z^2/2! + z^4/4! + \cdots$, we see that when a, b and c are very small, the formula is, up to the second-order terms, $c^2 = a^2 + b^2$. Thus, infinitesimally, the hyperbolic version of Pythagoras' theorem agrees with the Euclidean version. This is because the hyperbolic distance is obtained form the Euclidean distance by applying a 'local scaling factor' of $1/y$ at z. As this scaling factor is essentially constant on an infinitesimal neighbourhood of a point, the 'infinitesimal hyperbolic geometry is just a scaled version of the Euclidean geometry. However, as the scaling factor varies considerably over large distances, the global hyperbolic geometry is very different from the Euclidean geometry. For example, if in Pythagoras' theorem, a, b and c are all very large, then, as $\cosh x$ is approximately $e^x/2$ when x is large, we have (approximately) $4e^c = e^a e^b$ so that $c = a + b - \log 2$. In other words, in a 'large' hyperbolic right-angled triangle, the length of the hypotenuse is almost the sum of the lengths of the other two sides! If this were the case in Euclidean geometry, then the triangle would be very 'flat', but this is not so in hyperbolic geometry.

Finally, we remark that hyperbolic trigonometry is as rich and well understood as Euclidean trigonometry (and spherical trigonometry) is. For example, there is a sine rule, and cosine rules in hyperbolic geometry. In most applications it is the hyperbolic trigonometry that is important, and hyperbolic geometry by itself has relatively few applications.

Exercise 15.4

1. Suppose that a, b and c are the sides of a right-angled hyperbolic triangle with the right-angle opposite the side of length c. Prove that $c \leq a + b$; this is a special case of the triangle inequality.
2. Consider a right-angled hyperbolic triangle with both sides ending at the right angle having length a. Let the height of this triangle be h (the distance from the right angle to the third side). Find h as a function of a. What is the limiting behaviour of h as $a \to +\infty$?

15.5 Hyperbolic three-dimensional space

We end this text with a very brief description of three-dimensional hyperbolic geometry, and a sketch of the proof of Theorem 14.3.2. These are given in this and the next section, and they combine many of the ideas that have been introduced in this text. We take *hyperbolic space* to be the upper-half of $\mathbb{R}^3$, namely

$$\mathcal{H}^3 = \{(x, y, t) \in \mathbb{R}^3 : t > 0\}.$$

It is convenient to identify the point (x, y, t) with the quaternion $x + y\mathbf{i} + t\mathbf{j}$, and also to identify the quaternion $\mathbf{i}$ with the complex number i. Thus we can write (x, y, t) as $z + t\mathbf{j}$, where z is the complex number $x + iy$. Note that in this notation we have the convenient formula

$$z\mathbf{j} = (x + y\mathbf{i})\mathbf{j} = x\mathbf{j} + y\mathbf{k} = x\mathbf{j} - y\mathbf{j}\mathbf{i} = \mathbf{j}\bar{z}. \tag{15.5.1}$$

Suppose now that $g(z) = (az+b)/(cz+d)$, where $ad - bc \neq 0$. We can now let g act on hyperbolic space $\mathcal{H}^3$ by the rule

$$g : z + t\mathbf{j} \mapsto \left[a(z+t\mathbf{j}) + b\right]\left[c(z+t\mathbf{j}) + d\right]^{-1}, \tag{15.5.2}$$

where this computation is to be carried out in the algebra of quaternions. This lengthy (but elementary) exercise shows that

$$g(z+t\mathbf{j}) = \frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}t^2 + |ad-bc|t\mathbf{j}}{|cz+d|^2+|c|^2t^2}. \qquad (15.5.3)$$

Notice that as quaternions are not commutative, we have to choose (and then be consistent about) which side we shall write the inverse in (15.5.2). However, in (15.5.3), the denominator is real (and positive), and as every real number commutes with every quaternion, we can write it in the usual form for a fraction without any ambiguity. Notice also that if we put $t = 0$ in (15.5.3), we recapture the correct formula for the action of g on $\mathbb{C}$.

The consequences of (15.5.3) are far-reaching. First, if we consider g to be a translation, say $g(z) = z + b$, then we find that $g(z + t\mathbf{j}) = (z + b) + t\mathbf{j}$; thus g is just the 'horizontal' translation by b. If $g(z) = az$, then we find that

$$g(z+t\mathbf{j}) = az + |a|t\mathbf{j}.$$

If $|a| = 1$, so that g is a rotation of the complex plane, then g acts on $\mathcal{H}^3$ as a rotation about the vertical axis through the origin. If $a > 0$, so that g acts as a 'stretching' from the origin by a factor a in $\mathbb{C}$, then g also acts as a stretching (from the origin, and by the same factor) in $\mathcal{H}^3$. Of course, the more interesting case is when $g(z) = 1/z$; here

$$g(z+t\mathbf{j}) = \frac{\bar{z}+t\mathbf{j}}{|z|^2+t^2}.$$

We define the lines in $\mathcal{H}$ to be the 'vertical' semi-circles, and the 'vertical' rays (exactly as in the two-dimensional case; see Figure 15.5.1), and we can define the hyperbolic distance between two points again by a cross-ratio (thinking of the vertical plane through the two points as the complex plane), or by integrating $|dx|/x_3$, where $x = (x_1, x_2, x_3)$, over curves. When all this has been done, we arrive at the following beautiful result.

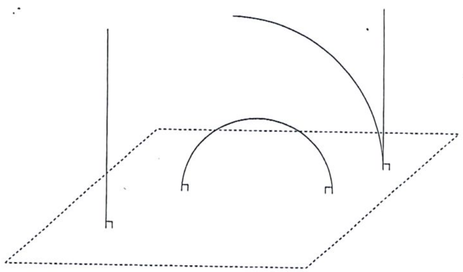


Figure 15.5.1

Theorem 15.5.1 *Every Möbius map acts on hyperbolic space $\mathcal{H}^3$ as a hyperbolic isometry, and every isometry that preserves orientation is a Möbius map.*

15.6 Finite Möbius groups

Finally, we give only the briefest sketch of the ideas behind a proof of Theorem 14.3.2. The aim of this sketch is to give the reader a glimpse of some beautiful interaction between algebra and geometry, and it is far from being complete. First, the Möbius maps (that act on $\mathbb{C}_\infty$) can be extended (either as a composition of reflections and inversions, or in terms of the quaternion algebra) to act on all of $\mathbb{R}^3$. The upper-half $\mathcal{H}^3$ of $\mathbb{R}^3$ with the hyperbolic metric $ds = |dx|/x_3$ is a model of three-dimensional hyperbolic geometry, and the Möbius group is the group of orientation-preserving isometries of this space.

Now let G be a finite Möbius group; then G may be regarded as a finite group of isometries of $\mathcal{H}^3$, so that each point in $\mathcal{H}^3$ has a finite orbit. Take any orbit and let B be the smallest hyperbolic ball that contains the orbit. Analytic arguments show that B is unique, and as the chosen orbit is invariant under G, so is B, and hence (finally) so too is the hyperbolic centre of B. This argument proves that the elements of the finite group G have a common fixed point ζ in $\mathcal{H}^3$. There is now a Möbius map (which acts on all of $\mathbb{R}^3 \cup \{\infty\}$) that converts the upper-half space model of three-dimesional hyperbolic space into the unit ball model (much as there is a Möbius map that takes the upper half-plane to the unit disc). This can be chosen so that ζ is carried to the origin; thus the finite Möbius G is conjugate to a Möbius group G' of hyperbolic isometries that act on the unit ball in $\mathbb{R}^3$ with the extra property that every element of G' fixes 0. It is not difficult to show that every such isometry is a Euclidean rotation of $\mathbb{R}^3$ and the sketch of the proof is complete. $\square$

Exercises on Hyperbolic Plane

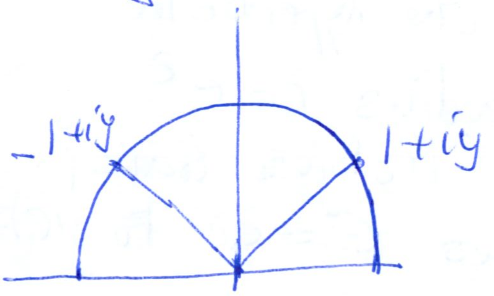
15.2.2 Consider the function $\sinh z = \frac{e^z - e^{-z}}{2}$ and $\cosh(z) = \frac{e^z + e^{-z}}{2}$

a) $\cosh^2 z - \sinh^2 z = \frac{e^{2z} + e^{-2z} - e^{2z} - e^{-2z} + 2 + 2}{4} = 1$

b) $\cosh(2z) = \frac{e^{2z} + e^{-2z}}{2} = 2\left(\frac{e^{2z} + e^{-2z} + 2 - 2}{4}\right) = 2\cosh^2 z - 1$

$$= 2\left(\frac{e^{2z} + e^{-2z} - 2 + 2}{4}\right) = 2\sinh^2 z + 1$$

15.2.3 Find the hyperbolic distance between $z = 1+iy$ and $\omega = -1+iy$ as a function of y. Show that for a given positive t there is a value of y s.t $\rho(1+iy, -1+iy) = t$.



We have that

$$\cosh^2 \tfrac{1}{2}\rho(z,\omega) = \frac{|1+yi-(-1+iy)|^2}{4y^2}$$

$$\cosh^2 \tfrac{1}{2}\rho(z,\omega) = \frac{1+y^2}{y^2}$$

By 15.2.2b) $\cosh \rho(z,\omega) = 2\cosh^2 \tfrac{1}{2}\rho(z,\omega) - 1$

$$= \frac{2+2y^2}{y^2} - 1 = \frac{2+y^2}{y^2}, \quad y>0$$

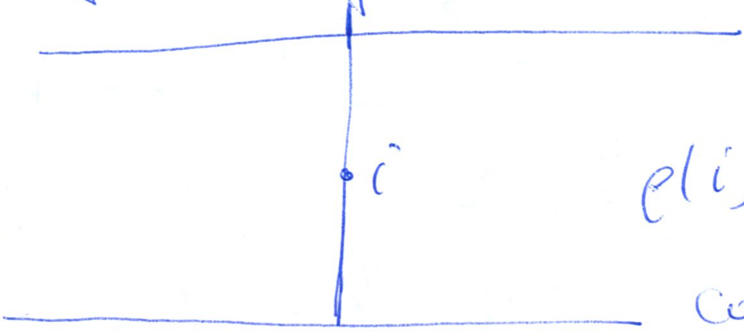
Now given $t > 0$ s.t $\frac{e^{2t}+1}{2e^t} = \frac{2+y^2}{y^2}$, $y>0$

$s = e^t$ satisfies $y^2 s^2 - 2(2+y^2)s + y^2 = 0$

or $y^2(e^t - 1)^2 = 4e^t$

so $y = \frac{2e^{\frac{1}{2}t}}{e^t - 1}$ (Observe that $t>0$, $e^t > 1$)

15.2.4 Consider the set $L \subseteq \mathcal{H}$ defined by the equation $L = \{ z \in \mathbb{C} ;\ \mathrm{Im}[z] = 2 \}$



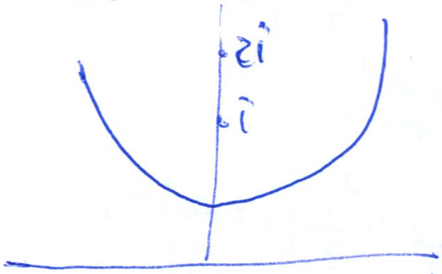
The distance $\rho(i, x+2i)$ satisfies that

$$\cosh \frac{1}{2} \rho(i, x+2i) = \frac{|-x+3i|}{8} = \frac{x^2+9}{8}$$

As $\cosh \frac{1}{2} \rho(i, x+2i)$ has the minimum for $x=0$ the distance $\rho(i, x+2i)$ has the minimum for $x=0$ i.e. $\underline{z = 2i}$ So

$$\rho(i, L) = \rho(i, 2i)$$

15.3.1 Give the equation of the hyperbolic circle with centre $2i$ and radius $r = e^2$



First of all the Mobius transf. $g(z) = \frac{1}{2} z$ takes $C = 2i$ to $g(C) = i$ and $g^{-1}(z) = 2z$

Now, we know that the equation for $g(C)$ is $x'^2 + (y' - \cosh e^2)^2 = \sinh^2(e^2)$ where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So $C = g^{-1}(gC)$ has the equation $\frac{1}{4} x^2 + \frac{1}{4}(y - 2\cosh e^2)^2 = \sinh^2(e^2)$

$$x^2 + (y - 2\cosh e^2)^2 = (2 \sinh(e^2))^2$$

For $x = 0$

$$(y - 2\cosh e^2)^2 = \begin{cases} 2\sinh(e^2) \\ -2\sinh(e^2) \end{cases}$$

$z_1 = 2(\cosh e^2 + \sinh e^2) i$

$z_2 = 2(\cosh e^2 - \sinh e^2) i$

15.3.2 We compare the equations for circles in Euclidean geometry, Spherical, Hyperbolic

E G

$$(x-a)^2+(y-b)^2=r^2$$

$$\boxed{x^2+y^2=r^2}$$

S G

$(\cos\lambda\cos\theta, \cos\lambda\sin\theta, \sin\lambda)$

(λ, θ) at distance

$\frac{\pi}{2}-\lambda$ to $N(0,0,1)$

$\frac{\pi}{2}-\lambda=r$

So $x^2+y^2+\cos^2 r=1$

or $\boxed{x^2+y^2=\sin^2 r}$

H G

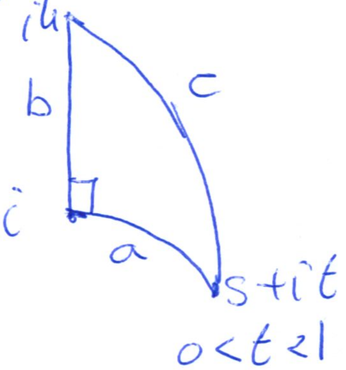
centre i, $z=x+iy$

$y>0$

$$\boxed{x^2+(y-\cosh r)^2=\sinh^2 r}$$

<u>15.4.1</u> Suppose that a, b, c are the sides of a right-angled triangle with the right-angle opposite the side of length c. Show that $c \leq a+b$.

First of all, we can assume that the triangle is



$0<t<1$

We know that

$$\cosh a = \frac{k^2+1}{k^2}$$

$$\cosh b = \frac{1}{t}$$

$$\cosh c = \frac{k^2+1}{2tk}$$

We know also that $\cosh(a+b)=\cosh a\cosh b+\sinh a\sinh b$

We see that $\cosh c \leq \cosh(a+b)$.

As $\cosh(x)$ is increasing on positive real numbers then $c \leq a+b$

1) Detailed Calculation hyperbolic distance between

$z = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

Method 1: Using $\sinh^2\left(\frac{1}{2} d_H\right) = \frac{|z-\omega|^2}{4 \operatorname{Im} z \operatorname{Im} \omega}$

we have $\sinh^2\left(\frac{1}{2} d\right) = \frac{\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)^2 + 0^2}{4 \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2}} = \frac{1}{3}$

$$\sinh\left(\frac{1}{2} d\right) = \frac{1}{\sqrt{3}} \; ; = \frac{\left(e^{\frac{1}{2}d}\right)^2 - 1}{2 e^{\frac{1}{2}d}} \quad \left(\text{change of var } e^{\frac{1}{2}d} = y\right)$$

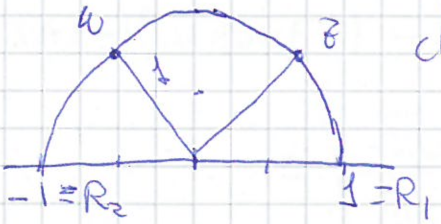
$$\frac{2}{\sqrt{3}} y = y^2 - 1 \; ; \; y^2 - \frac{2}{\sqrt{3}} y - 1 = 0$$

$$y = \frac{\frac{2}{\sqrt{3}} \pm \sqrt{\frac{4}{3} + 4}}{2} = \frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}} \Rightarrow y = \sqrt{3}$$

$$e^{\frac{1}{2}d} = \cancel{1/}\frac{1}{\sqrt{3}} \; ; \; \frac{1}{2} d = \left|\ln \frac{1}{\sqrt{3}}\right| \; ; d = 2 \ln \sqrt{3}$$

$$d = \ln 3$$

Method 2: Using the cross-ratio, z and ω belong to the Euclidean unit circumference



$$d_H(z,\omega) = |\ln R(z, \omega, R_2, R_1)| =$$

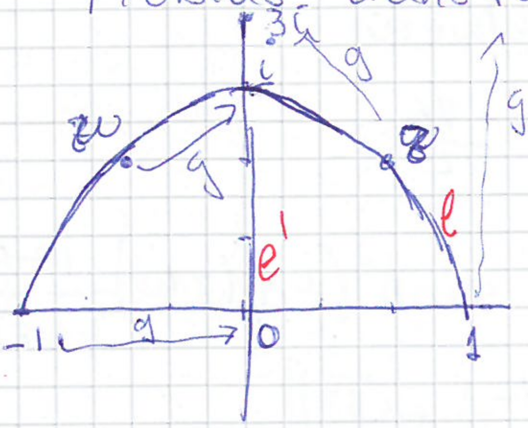
$$= \left|\ln R\left(\frac{1}{2} + \frac{\sqrt{3} i}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -1, 1\right)\right|$$

$$\cancel{= \left|\ln \frac{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)}{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)}\right| = \left|\ln\left(-(1+\sqrt{3})(1+\sqrt{3})\right.\right.}$$

$$= \left|\ln \frac{\left(-\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)\left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)}{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)}\right| = \left|\ln \frac{-3(\sqrt{3}+i)(\sqrt{3}-i)}{(1+i\sqrt{3})(1-i\sqrt{3})}\right|$$

$$= \left|\ln\left(-3 \frac{4}{-4}\right)\right| = |\ln 3| = \ln 3$$

Method 3 Mapping the hyperbolic line containing z and w to the imaginary axis with a Mobius transformation



$$g(z) = \frac{az+b}{cz+d}$$

$$\begin{cases} g(-1) = 0 \Rightarrow -a+b=0 \\ g(1) = \infty \Rightarrow c+d=0 \\ ad-bc = 1 \,;\ 2ad=1 \\ b=a,\ d=\frac{1}{2a},\ c=\frac{-1}{2a} \end{cases}$$

$$g\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = i$$

$$\Leftrightarrow \frac{a\left(\frac{-1}{2}+\frac{i\sqrt{3}}{2}+1\right)}{\frac{1}{2a}\left(-\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)+1\right)} = i$$

$$i = \frac{2a^2\,\frac{1}{2}(1+i\sqrt{3})}{\frac{1}{2}(3-i\sqrt{3})}\ ;$$

$$a^2 = \frac{\sqrt{3}}{2}\, i\, \frac{(\sqrt{3}-i)}{(1+i\sqrt{3})} = \frac{\sqrt{3}}{2}\, i\, \frac{(\sqrt{3}-i)(1-i\sqrt{3})}{4}$$

$$= \frac{\sqrt{3}}{2} \quad ; \quad a = \sqrt{\frac{\sqrt{3}}{2}}$$

$$g(z) = 2a^2\left(\frac{z+1}{-z+1}\right) = \sqrt{3}\left(\frac{z+1}{-z+1}\right)$$

$$g\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = \sqrt{3}\left(\frac{\frac{3}{2}+\frac{i\sqrt{3}}{2}}{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) = \frac{3(\sqrt{3}+i)(1+i\sqrt{3})}{4} = 3i$$

$$d_H(z,w) = d_H(w,z) = \left|\ln R(i, 3i, 0, \infty)\right| = \left|\ln\left(\frac{-i}{-i3}\right)\right| = \left|-\ln 3\right|$$

$$= \ln 3$$

Obs: We can use the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, but then we need bases for the projective lines ℓ (containing z and w) and ℓ' (imaginary axis)

2) Calculate the hyperbolic distance between

$z = \frac{1+i}{8}$, and $w = \frac{7+i}{8}$

<u>Method 1</u> $\sinh^2\left(\frac{1}{2} d_H\right) = \dfrac{\left|-6/8\right|^2}{4 \cdot \frac{1}{8} \cdot \frac{1}{8}} = 9,$

$$\sinh^{\cancel{2}}\left(\tfrac{1}{2} d\right) = 3$$

$$\sinh\left(\tfrac{1}{2} d\right) = \frac{\left(e^{\frac{1}{2}d}\right)^2 - 1}{2\, e^{\frac{1}{2}d}} = 3, \quad \left(y = e^{\frac{1}{2}d}\right)$$

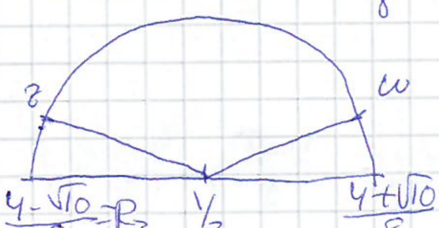
$$6y = y^2 - 1 \; ; \quad y^2 - 6y - 1 = 0$$

$$y = \frac{6 \pm \sqrt{36+4}}{2} = 3 \pm \sqrt{10}, \quad y = 3 + \sqrt{10}$$

$$\tfrac{1}{2} d = \ln(3+\sqrt{10}), \quad d = 2\ln(3+\sqrt{10})$$

$$d = \ln(3+\sqrt{10})^2 = \ln(19 + 6\sqrt{10})$$

<u>Method 2</u> the hyperbolic line through z and w is the half circumference with centre $z_c = \frac{1}{2}$ and radius $r = \frac{\sqrt{10}}{8}$



$$d_H(z, w) = \left|\ln R\left(\frac{7+i}{8}, \frac{1+i}{8}, \frac{4-\sqrt{10}}{8}, \frac{4+\sqrt{10}}{8}\right)\right|$$

$$= \left|\ln \frac{(-3-\sqrt{10}-i)(3+\sqrt{10}-i)}{(3-\sqrt{10}-i)(-3+\sqrt{10}-i)}\right| =$$

$$= \left|\ln \frac{(3+\sqrt{10}+i)(3+\sqrt{10}-i)}{(-3+\sqrt{10}+i)(-3+\sqrt{10}-i)}\right| =$$

$$= \left|\ln \frac{(3+\sqrt{10})^2+1}{(-3+\sqrt{10})^2+1}\right| = \left|\ln \frac{1+9+10+6\sqrt{10}}{1+9+10-6\sqrt{10}}\right| = \left|\ln \frac{\sqrt{10}+3}{\sqrt{10}-3}\right|$$

$$= \left|\ln -\frac{(\sqrt{10}+3)^2}{9-10}\right| = \left|\ln(\sqrt{10}+3)^2\right| = 2\ln(\sqrt{10}+3)$$

$$= \ln(19 + 6\sqrt{10})$$

Example 4 $\gamma(t) = (t, 0, t^2)$ is a geodesic on $x(x,y) = (x, y, x^2-y^2)$.

$x_1 = (1, 0, 2x)$ $N(\gamma) = (-2t, 0, 1)\frac{1}{\sqrt{1+4t^2}}$ // osculating plane.

$x_2 = (0, 1, -2y)$ $y=0$

Theorem Let $x: \mathcal{U} \to W \cap S$ be a chart for S with first fundamental form $I_x = (g_{ij})$. The following holds:

$$\Gamma_{ij}^k = \frac{1}{2} \sum g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} + \frac{\partial g_{lj}}{\partial x_i} \right)$$

Proof $\Gamma_{ij}^k = \sum g^{kl} (x_{ij} \cdot x_l)$ and

$$\left.\begin{aligned} \frac{\partial g_{il}}{\partial x_j} &= x_{ij} \cdot x_l + x_i \cdot x_{jl} \\ \frac{\partial g_{ij}}{\partial x_l} &= x_{il} \cdot x_j + x_i \cdot x_{jl} \\ \frac{\partial g_{jl}}{\partial x_i} &= x_{ij} \cdot x_l + x_j \cdot x_{il} \end{aligned}\right\} \text{adding } \Gamma_{ij}^k = \sum_l g^{kl}(x_{ij} \cdot x_l) = \frac{1}{2} \sum g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} + \frac{\partial g_{jl}}{\partial x_i} \right)$$

Corollary 1 Geodesics only depend on I_x

Corollary 2 An isometry transforms geodesics to geodesics

Theorem Let P be a point on a surface S and let u_0 be a unit tangent vector ~~to~~ S at P. Then there is a unique geodesic γ through P with tangent vector u_0 at P.

Proof γ is the integral curve of the system with initial conditions
$$\begin{cases} \frac{d^2 x_k}{ds^2} + \sum \Gamma_{ij}^k \frac{dx_1}{ds} \frac{dx_2}{ds} = 0, \; k=1,2 \\ \gamma(0) = P \quad \dot{\gamma}|_0 = u_0 \end{cases}$$

Example (Surfaces of revolution)

$\mathbf{x}(s,\theta) = (f(s)\cos\theta,\ f(s)\sin\theta,\ g(s))$; $I_{\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 0 & f^2 \end{pmatrix}$

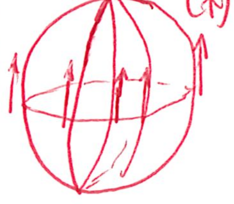
$\mathbf{x}_1 = (f'\cos\theta,\ f'\sin\theta,\ g')$ $\quad \mathbf{x}_{11} = (f''\cos\theta,\ f''\sin\theta,\ g'')$

$\mathbf{x}_2 = (-f\sin\theta,\ f\cos\theta,\ 0)$ $\quad \mathbf{x}_{12} = (-f'\sin\theta,\ f'\cos\theta,\ 0)$

$\mathbf{x}_{22} = (-f\cos\theta,\ -f\sin\theta,\ 0)$

$$\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/f^2 \end{pmatrix} \begin{pmatrix} \overset{f'f''+g'g''}{0} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/f^2 \end{pmatrix} \begin{pmatrix} 0 \\ f f' \end{pmatrix} = \begin{pmatrix} 0 \\ f'/f \end{pmatrix}$$

Image of (*)



$$\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/f^2 \end{pmatrix} \begin{pmatrix} -f f' \\ 0 \end{pmatrix} = \begin{pmatrix} -f f' \\ 0 \end{pmatrix}.$$ Equations:

$$\begin{cases} \cancel{d}s'' - f f' \theta'^2 = 0 \\ \theta'' + f'/f\, \theta' s' = 0 \end{cases}$$

$\Longrightarrow$ i) Meridians ($\theta' = \theta'' = 0$) are geodesics.

ii) Parallels are geodesics if $f' = 0$ for $s = s_0$; it means that the tangent to the meridians is parallel to the axis of revolution along the parallel. (*)

Example: Determine the geodesics on the Poincaré hyperbolic plane $\mathbb{H}$ (upper half plane) with first fundamental form $I_{\mathbf{x}} = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$ $\left(ds^2 = \frac{1}{y^2}(dx^2 + dy^2)\right)$; $I_{\mathbf{x}}^{-1} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$

$$\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial \cancel{\mathbf{x}}} - \frac{\partial g_{11}}{\partial \cancel{\mathbf{x}}} + \frac{\partial g_{11}}{\partial \cancel{\mathbf{x}}}\right) + \frac{1}{2} g^{12}(\quad) \\ \frac{1}{2} g^{12}\left(\quad - \frac{\partial g_{11}}{\partial x}\quad\right) + \frac{1}{2} g^{22}\left(\frac{\partial g_{12}}{\partial y} - \frac{\partial g_{11}}{\partial y} + \frac{\partial g_{12}}{\partial y}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2}\frac{2}{y^3} y^2 \end{pmatrix}$$

$\Gamma_{11}^1 = 0,\ \Gamma_{11}^2 = -1/y,\ \Gamma_{12}^1 = -1/y,\ \Gamma_{12}^2 = 0;\ \Gamma_{22}^1 = 0;\ \Gamma_{22}^2 = -1/y$

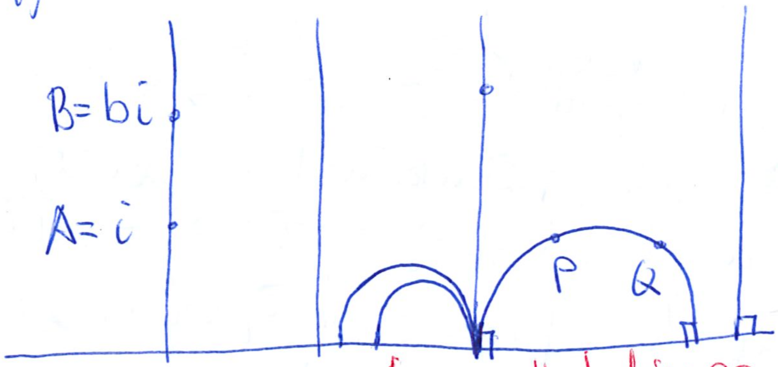
$$\begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial y} - \quad\right) + \frac{1}{2} g^{12}(\quad) \\ \frac{1}{2} g^{21}\left(\frac{\partial g_{11}}{\partial y} - \quad\right) + \frac{1}{2} g^{22}(\quad) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\frac{1}{y^2}\frac{2}{y^3} \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g^{11}\left(\frac{\partial g_{12}}{\partial x} - \frac{\partial g_{22}}{\partial x} + \frac{\quad}{\partial x}\right) + g^{12}(\quad) \\ g^{12}(\quad) + g^{22}\left(\frac{\partial g_{22}}{\partial y}\right) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ g^{22}\frac{\partial g_{22}}{\partial y} \end{pmatrix}$$

Equations $\begin{cases} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 \\ \ddot{y} - \frac{1}{y}(\dot{x}^2 + \dot{y}^2) = 0 \end{cases}$ Integral curves $\begin{cases} x = a \\ \text{or} \\ (x-a)^2 + y^2 = a^2 \end{cases}$

i) Geodesics in $\mathbb{H}$ Cline at infinity — $\mathbb{R} \cup \{\infty\}$

There is a unique geodesic passing two points



4 parallel lines in $\mathbb{H}$

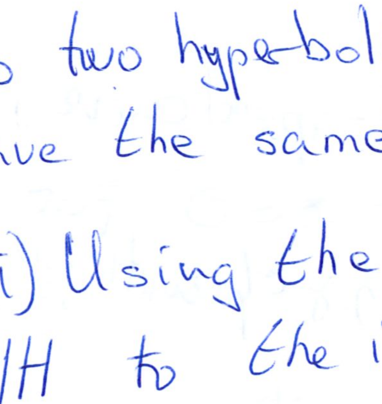
$$d_{\mathbb{H}}(P,Q) = 2\,\operatorname{arctanh}\frac{|Q-P|}{|Q-\bar{P}|}$$, where $P, Q \in \mathbb{C}$, $\operatorname{Im}P, \operatorname{Im}Q > 0$

ii) Let $T = \triangle ABC$ be a hyperbolic triangle

$$Ar(T) = \pi - \alpha - \beta - \gamma$$

So two hyperbolic triangles are congruent iff they have the same angles.

iii) Using the Mobius $f(z) = \frac{z-i}{z+i}$ one transforms $\mathbb{H}$ to the interior of the unit disk $\mathbb{D}$, this gives us another model of ~~the~~ the hyperbolic plane: Poincaré's Unit Disk. The geodesics are, by mapping the geodesic above ~~to~~ under $f(z) = \frac{z-i}{z+i}$, the diametres and half-circles orthogonal to $\{z; |z|^2=1\}$



Cline at infinity $\{z; |z|^2 = 1\}$

the metric is given by

$$I_{\mathbb{D}} = \begin{pmatrix} \frac{4}{1-v^2w^2} & 0 \\ 0 & \frac{4}{1-v^2w^2} \end{pmatrix} ; |v+iw|^2 < 1$$

$I_x = \begin{pmatrix} 1 & 0 \\ 0 & g_{22} \end{pmatrix}$. Then $K = \frac{-1}{\sqrt{g_{22}}} \left(\frac{\partial^2 \sqrt{g_{22}}}{\partial x_1^2} \right)$.

We have the following result

Th (Minding) Any two regular surfaces with the same constant Gaussian curvature are locally isometric.

Observation (~~the~~ a consequence a point on a surface of constant Gaussian curvature belongs to a chart that is (a piece of) the plane, the sphere or the ~~pseudos~~ pseudosphere = hyperbolic plane

Proof: Use geodesic coordinates $I_x = \begin{pmatrix} 1 & 0 \\ 0 & g_{22} \end{pmatrix}$

We have that $k = -\frac{1}{\sqrt{g_{22}}} \left(\frac{\partial^2 \sqrt{g_{22}}}{\partial x_1^2} \right)$

Case 1) If $k=0$ $\frac{\partial^2 \sqrt{g_{22}}}{\partial x_1^2} = 0$ and $\sqrt{g_{22}} = f(x_2)$,

since $\lim_{x_1 \to 0} \left(\frac{\partial \sqrt{g_{22}}}{\partial x_1} \right) = 1$, $\frac{\partial \sqrt{g_{22}}}{\partial x_1} = 1$. Thus $\sqrt{g_{22}} = x_1 + f(x_2)$

Since $\lim_{x_1 \to 0} \sqrt{g_{22}} = 0 = f(x_2) \Rightarrow g_{22} = x_1^2$

$I_x = \begin{pmatrix} 1 & 0 \\ 0 & x_1^2 \end{pmatrix}$ which is the I-fundamental form of the plane in polar coordinates $\begin{cases} x_1 = \rho \cos \theta \\ x_2 = \rho \sin \theta \end{cases}$

Case 2 If $k > 0$, the general solution is given by

$\sqrt{g_{22}} = A(x_2) \cos(\sqrt{k}\, x_1) + B(x_2) \sin(\sqrt{k}\, x_1)$

As $\lim_{x_1 \to 0} \sqrt{g_{22}} = 0$, ~~then~~ $A(x_2) = 0$, $\frac{\partial \sqrt{g_{22}}}{\partial x_1} = B(x_2)\sqrt{k} \cos(\sqrt{k}\, x_1)$

Again, since $\lim_{x_1 \to 0} \frac{\partial \sqrt{g_{22}}}{\partial x_1} = 1$, we obtain $B(x_2) = \frac{1}{\sqrt{k}}$

$$g_{22} = \frac{1}{k} \sin^2(\sqrt{k}\, x_1)$$

Now $I_x = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sin^2(\sqrt{k}\,x_1)}{k} \end{pmatrix}$; the I-fundamental form of a sphere of radius $\frac{1}{\sqrt{k}}$.

Case 3 If $k<0$. The general solution of the equation $\sqrt{g_{22}}\,k + \frac{\partial^2 \sqrt{g_{22}}}{\partial x_1^2} = 0$ is

$$\sqrt{g_{22}} = A(x_2)\cosh(\sqrt{-k}\,x_1) + B(x_2)\sinh(\sqrt{-k}\,x_1)$$

As in the previous cases $A(x_2)=0$ and $B(x_2) = \frac{1}{\sqrt{-k}}$; $g_{22} = \frac{1}{-k}\sinh^2(\sqrt{-k}\,x_1)$

which the I-fundamental form of a pseudosphere generated by a tractrix of length of the chain $\frac{1}{\sqrt{-k}}$.

As a consequence we have the following characterization of the sphere

Th (The rigidity of the sphere) Let S be a compact, connected, regular surface with constant Gaussian curvature k. Then S is a sphere

Observation: this theorem says more than Minding's Th. The surface is not only locally isometric to S^2, is S^2 (if $k=1$, otherwise change the radius)

As the surface is compact $k>0$.
The first proof is due to Liebmann (1899). But the proof based in the following result is

from Hilbert and Chern