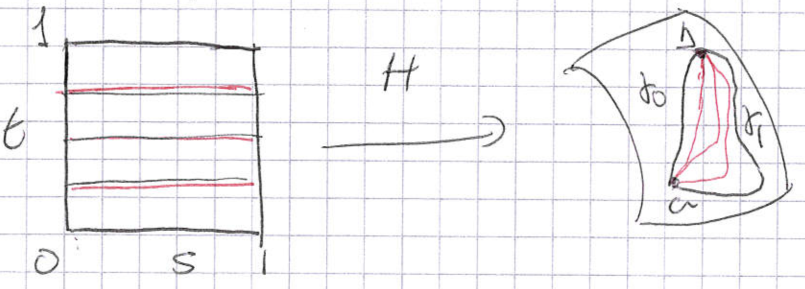
Simply connected domains in $\mathbb{C}$

Consider a topological space X. We say that two paths γ_0 and γ_1 are <u>homotopic</u> if there exists a continuous function

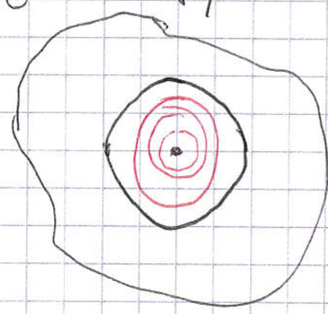
$$H : I \times I \longrightarrow X$$

$H(s,0) = \gamma_0(s)$, $H(s,1) = \gamma_1(s)$, (if $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$, also $H(0,t) = a$, $H(1,t) = b$, $\forall s,t \in I^2$

We say that H is a <u>homotopy from γ_0 to γ_1</u>



<u>Example</u> Consider $\gamma_0 : I \longrightarrow \mathbb{C}$ $\gamma_0(s) = 0$ $\forall s$ and $\gamma_1 : I \longrightarrow \mathbb{C}$ $\gamma_1(s) = e^{2\pi i s}$. γ_0 and γ_1 homotopic in $\mathbb{C}$



$$H(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s)$$

But γ_0 and γ_1 are not homotopic in $\mathbb{C} \setminus \{1/2\}$

<u>Def</u> A domain D of $\mathbb{C}$ is said to be <u>simply connected</u> if any closed path γ is homotopic to a constant path $c \in D$ in D. We say that γ is null-homotopic in D

We have the following characterizations :

Th Let D be a domain of $\mathbb{C}$. The following properties are equivalent

i) D homeomorphic to $\mathbb{D}$ (unit disc)

ii) D simply connected

iii) Let $Ind_\gamma(a) = 0$, $\forall$ closed path $\gamma \subseteq D$ and $a \in \mathbb{C} \setminus D$
($Ind_\gamma(a)$ is the rotation index ~~of~~ of γ around a)

iv) $\hat{\mathbb{C}} \setminus D$ connected

v) Given f analytic on D, f can be uniformly approximated on compact subsets K of D by polynomials

vi) For each f analytic on D and each closed path $\gamma \subseteq D$

$$\int_\gamma f(z)\,dz = 0$$

vii) Given f analytic on D there is F analytic on D

$$F' = f$$

viii) If both f and $1/f$ analytic on D, then there is an analytic function g on D s.t $f = e^g$

ix) If both f and $1/f$ analytic on D, there is g analytic on D s. t $f = g^2$

Def Let D_1 and D_2 be domains in $\mathbb{C}$. We say that D_1 and D_2 are conformally equivalent if there exists a bijective conformal (analytic) function $f: D_1 \to D_2$

Th (Riemann) Given a simply connected domain $D \subseteq \mathbb{C}$. Either $D = \mathbb{C}$ or D conformally equivalent to $\mathbb{D}$.

Proof First of all, by Liouville's Th $\mathbb{C}$ and $\mathbb{D}$ are not conformally equivalent

Now consider $D \neq \mathbb{C}$ domain; there exists $w_0 \in \mathbb{C} \setminus D$

Consider $\Sigma = \{\varphi : D \xrightarrow{1-1} \mathbb{D} \text{ analytic}\}$. We have to see that there is $\underline{\varphi \in \Sigma \text{ bijective}}$

Again take $w_0 \in \mathbb{C} \setminus D$ (such a w_0 does exist since $D \subsetneq \mathbb{C}$). Let $f : D \to \mathbb{C}$ defined by $f(z) = z - w_0$, as f and $1/f$ analytic on D, there is φ analytic on D s.t $\varphi^2 = f = z - w_0$. Observe that φ injective on D, moreover there are <u>no</u> two points $z_1 \neq z_2$ in D. s.t $\varphi(z_1) = -\varphi(z_2)$.

Since φ analytic, then φ open, so there is a disc $D(a,r) \subseteq \varphi(D)$ s.t $D(-a,r) \cap \varphi(D) = \emptyset$
$(0 < r < |a|)$

We can define $\psi(z) = \frac{r}{\varphi(z)+a}$ where $\psi \in \Sigma$, i.e $\underline{\Sigma \neq \emptyset}$

Finally if given an injective funct $\psi \in \Sigma$, not surjective then there exists $z_0 \in D$ and $\psi_1 \in \Sigma$ satisfying $|\psi_1'(z_0)| > |\psi'(z_0)|$, what allows us to find a bijective analytic funct. $h : D \to \mathbb{D}$.

Consider now the ~~meromorphic~~ analytic bijective functions

$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha} z} \quad \text{for } \alpha \in \mathbb{D}$$

Let ψ be ~~in~~ Σ not surjective on $\mathbb{D}$, so $\exists \alpha \in \mathbb{D}$ s.t $\alpha \notin \psi(D)$, $\varphi_\alpha \circ \psi \in \Sigma$ and $\varphi_\alpha \circ \psi$ has <u>no</u> 0's on D, so there exists g analytic on D s.t $g^2 = \varphi_\alpha \circ \psi$

As above g is injective and considering $\beta = g(z_0)$ with $z_0 \in D$ $\psi_1 = \varphi_\beta \circ g \in \Sigma$, writing $\omega^2 = s(\omega)$ we have $\psi = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \circ \psi_1$.

Since $\psi_1(0) = z_0$, chain rule yields

$$\psi'(z_0) = F'(0)\, \psi_1'(z_0)$$

$F = \varphi_\alpha s \circ \varphi_{-\beta}$, with $F(\mathbb{D}) \subseteq \mathbb{D}$, not injective

Then (Schwarz) $|F'(0)| < 1$ and $|\varphi_1'(z_0)| > |\varphi'(z_0)|$

Given $z_0 \in D$ and consider $\underline{\eta = \sup\{|\varphi'(z_0)|, \varphi \in \Sigma\} > 0}$

By the result above if $h \in \Sigma$ is such that $|h'(z_0)| = \eta$ then h surjective.

Finally we find/construct such a function $h \in \Sigma$

As for any $\varphi \in \Sigma$ $|\varphi(z)| < 1$ then any sequence of functions in Σ contains a subsequence $\{\varphi_n\}$ that converges uniformly in compacts of D (Σ is called a normal family)

So as η is a sup there is a sequence $\{\varphi_n\}$ of functions in Σ satisfying that
$|\varphi_n'(z_0)| \to \eta$, we can consider that a subsequence $\{\varphi_n\}$ converges uniformly in compacts of D to h analytic on D with $\underline{|h'(z_0)| = \eta}$
h is <u>non-constant</u>, and $h(D) \subseteq \overline{\mathbb{D}}$ but in fact
$h(D) \subseteq \mathbb{D}$. ($h$ is surjective)

Using that h is the limit of $\{\varphi_n\}$ and that if $\alpha = h(z_1)$ and the 0's of $h - \alpha$ has no accumulation pt on D consider $z_2 \neq z_1$ and an open disc $B(z_2)$ of z_2, then $h - \alpha$ has no 0's on $B(z_2)$ and $h(z_2) \neq h(z_1)$.

$\underline{h : D \xrightarrow{\text{bijective}} \mathbb{D} \quad \text{analytic}}$