Chapter 0 The Riemann Space

Consider the set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the homeomorphisms

$$\varphi : \mathbb{C} \to S^2 \setminus \{N\}, \quad x+iy = z \mapsto \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

$$\psi : S^2 \setminus \{N\} \to \mathbb{C}, \quad (x_1, x_2, x_3) \mapsto \frac{x_1 + i x_2}{1 - x_3}$$

One can extend them continuously by $\varphi(\infty) = N$, $\psi(N) = \infty$

An open set $U \subseteq \hat{\mathbb{C}}$ is either U open in $\mathbb{C}$ or $U = (\mathbb{C} \setminus K) \cup \{\infty\}$, K compact in $\mathbb{C}$

Consider $\psi_S : S^2 \setminus \{S\} \to \mathbb{C}$, $\psi_S(x_1, x_2, x_3) = \frac{x_1 + i x_2}{x_3 + 1}$?

(and $\varphi_S(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right)$.

Clearly $\varphi_N^{-1} \varphi_S : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is $z \mapsto 1/z$ analytic

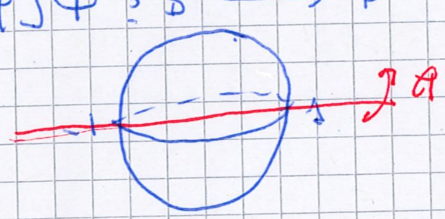
then ~~te~~ S^2 (or equivalently $\hat{\mathbb{C}}$) is a Riemann surface: the Riemann Sphere.

To see that $\hat{\mathbb{C}} = \mathbb{P}^1$ (projective line) we have ~~to~~ consider $U_0 = \{[z_0, z_1] \mid z_0 \neq 0\}$ and $U_1 = \{[z_0, z_1] \mid z_1 \neq 0\}$ the affine covering of $\mathbb{P}^1$ and the transition map on $U_0 \cap U_1$ $[z_0 : z_1] \underset{z_0 \neq 0}{=} [1 : z_1/z_0] = [z_0/z_1 : 1]$

If we consider $\varphi, \psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ we have the function

$$J : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$
$$z \mapsto 1/z \quad z \neq 0, z \in \mathbb{C}$$
$$0 \mapsto \infty$$
$$\infty \mapsto 0$$

$J^2 = \mathrm{Id}$; $\varphi J \psi : S^2 \to S^2$



We extend the concept of analytic and meromorphic at ∞.

Let D be a region (neighbourhood of ∞) f defined on $D \setminus \{\infty\}$

we should have $f(\infty) = \lim_{z \to 0} f(z^{-1}) = \lim_{z \to 0} (f \circ J)(z)$

f is analytic, meromorphic, etc at ∞ if $f \circ J$ is that at 0

Example i) $f(z) = \frac{1}{(z-1)^2}$, $f \circ J(z) = \frac{z^2}{(1-z)^2}$ is analytic at ∞, with a zero of order 2 at ∞

ii) $f(z) = z^k$, is analytic with a zero of order k ($k \leq -1$) at ∞, and is meromorphic with a pole of order k ($k \geq 1$) at ∞.

All meromorphic functions on $\vec{\mathbb{C}}$ are continuous and they form a group

As for the case of Laurent series of zeros and poles of an ~~analytic~~ meromorphic function on $\mathbb{C}$ we say that for $a = \infty$ is a solution of $f(\infty) = c$ with multiplicity k if $(f \circ J)(0) = c$ with multiplicity k. f has a pole of order k at ∞ ~~it~~ then $f(z) = \sum_{-\infty}^{k} a_j z^j$ near $z = \infty$, $a_k \neq 0$, again $\sum_{1}^{k} a_j z^j$ is the principal part

Consequence: A non-constant meromorphic function $f: \vec{\mathbb{C}} \to \vec{\mathbb{C}}$ takes any given value $c \in \vec{\mathbb{C}}$ only finitely many times, counting multiplicities.

To see it for $c = \infty$, the poles of f are the zeros of $f \circ J$ which are isolated. For $c \neq \infty$, we consider $f - c$, as $\vec{\mathbb{C}}$ compact so $f^{-1}(c) = \{z_1, \dots, z_s\}$ each with finite multiplicity.

Th i) Let f and g be meromorphic functions on $\hat{\mathbb{C}}$ with poles at the same pts and with the same principal parts at these pts. Then $f(z) = g(z) + c$

ii) Let f and g be meromorphic functions on $\vec{\mathbb{C}}$ with zeros and poles of the same orders at the same pts. Then $f(z) = c\,g(z)$

Proof ii) Consider f/g and g/f with no poles in $\mathbb{C}$, one is finite at ∞. $c \neq 0$

and then analytic, by Liouville's Th it is constant $c \neq 0$
$f/g = c$ $\therefore$ $\underline{f = gc}$

<u>Def</u> A rational function is a function of the form $f(z) = p(z)/q(z)$, p, q polynomials (complex coeff) not identically 0. When $z \in \mathbb{C}$, $q(z) \neq 0$, $f(z)$ well-defined, $q(z) = 0$ or $z = \infty$ $f(z) = \lim_{z' \to z} f(z')$. (We can consider that p, q coprime)

Rational functions form a field (the fractional field of $\mathbb{P}^1$): We denote it by $\mathbb{C}(z)$

We can express every rational function
$$f(z) = c(z-\alpha_1)^{m_1} \cdots (z-\alpha_r)^{m_r} (z-\beta_1)^{-n_1} \cdots (z-\beta_s)^{-n_s}$$
$\alpha_1, \dots \alpha_r$ the zeros of p (mult. m_i), $\beta_1, \dots \beta_s$ zeros of q (mult n_j). Then $\alpha_1, \dots \alpha_r$ zeros of f, mult. m_i and $\beta_1 \dots \beta_s$ poles of f, mult. n_j. Only zeros and poles of f on $\mathbb{C}$. As for ∞, ∞ is a zero or a pole as $(m_1 + \dots + m_r) - (n_1 + \dots + n_s)$ is negative or positive

Example i) $f(z) = \frac{1}{(z-1)^2}$, it has a zero of mult 2 at ∞ and a pole of mult. 2 at $z = 1$
ii) $f(z) = \frac{z^2}{z^2+1}$ has a zero of order 2 at $z = 0$, and simple poles at $z = i$ and $z = -i$.

<u>Th</u> A function $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational iff it is meromorphic

<u>Proof</u> $\Rightarrow$ Clear

$\Leftarrow$ Let f be meromorphic on $\hat{\mathbb{C}}$, f has finitely many poles in $\mathbb{C}$ say $\beta_1, \dots \beta_s$ of orders $n_1, \dots n_s$ and the function $g(z) = (z-\beta_1)^{n_1} \cdots (z-\beta_s)^{n_s} f(z)$ analytic on $\mathbb{C}$
$g(z) = a_0 + a_1 z + \dots$ $\forall z \in \mathbb{C}$ So g meromorphic at ∞
$(g \circ J)(z) = a_0 + a_1 z^{-1} + \dots$ meromorphic at 0 and $a_j = 0$ $\forall j \geq j_0$

$g(z)$ is a polynomial $f(z) = \dfrac{g(z)}{(z-\beta_1)^{n_1} \cdots (z-\beta_s)^{n_s}}$

<u>Def</u> Let f ~~be~~ p/q be a rational function (p, q coprime)

Then <u>degree</u> of f is the <u>maximum deg</u> of p and q

$\deg(f) = 0$ iff f constant

<u>Th.</u> If $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational function of deg $d > 0$, then f takes each value $c \in \hat{\mathbb{C}}$ exactly d times, counting mult.

<u>Proof</u> Let $f = p/q$, p and q co-prime polynomials.

First $c = \infty$: for $z \in \mathbb{C}$ $f(z) = \infty$ iff $q(z) = 0$ with $\deg q$ sol

If $\deg(p) \leq \deg q = d$ the only poles of f. If $d = \deg p > \deg q$

then f has a pole of order $d - \deg q$ at ∞, so in total d sol. for $f(z) = \infty$.

Now ~~for~~ $c \neq \infty$; since $\deg f > 0$, $f \not\equiv c$ and the rational function $g = \dfrac{1}{f-c} = \dfrac{q}{p-qc}$ has poles exactly at the sol of $f(z) = c$, but $\deg g = \deg f$ and g has exactly d poles counting multiplicity, as sol. of $f(z) = c$.

<u>Corollary</u> Let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational function of $\deg f = d > 0$. Then

i) f has only finitely many multiple pts.

ii) $|f^{-1}(c)| = d$ for all but finitely many pt $c \in \hat{\mathbb{C}}$, and $1 \leq |f^{-1}(c)| < d$ for the remaining pts.

Let f meromorphic at a pt a we have a valuation (order) of f at a: $v_a f \begin{cases} k & \text{if } f \text{ has a zero of mult } k \text{ at } a \\ 0 & \text{if } f(a) \neq 0, \infty \\ -k & \text{if } f \text{ has a pole of mult } k \text{ at } a \end{cases}$

(For $a \in \mathbb{C}$ $v_a(f)$ = lowest exp in Laurent series

$a = \infty$ $v_\infty(f)$ = lowest exponent of Laurent series $f(w)$ at 0

$$f(z) = c \prod_{a \in \mathbb{C}} (z-a)^{v_a(f)} \quad c \in \mathbb{C} \setminus \{0\}$$

$$v_\infty(f) = \deg q - \deg p = -\sum_{a \in \mathbb{C}} v_a(f), \text{ so } \boxed{\sum_{a \in \hat{\mathbb{C}}} v_a(f) = 0}$$

Let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a meromorphic, non-constant function.
If $f(a)=c$ with mult. k where $a, c \in \mathbb{C}$, f is k-to-1 near a. Furthermore if U is any sufficiently small open set containing a, then there exists an open set V containing c s.t for each $c' \in V \setminus \{c\}$ $f(z)=c'$ has k solutions in U, all simple. It is easy to see that any non-constant meromorphic function is open (extension of the result for analytic functions $f: D \to \mathbb{C}$)
If $f(a)=c$ with mult 1 then f maps $W = U \cap f^{-1}(V)$, neigh near a to V of c bijectively, then we have $f^{-1}: V \to W$ as ~~open~~, so f is open, is f^{-1} cont, as it is f so $f: W \xrightarrow{\text{homeo}} V$.
The set B of branch-points is finite and for any $c \in \hat{\mathbb{C}} \setminus f(B)$, $f(z)=c$ has only simple solutions: $a_1, \ldots, a_d \in$ $\in \hat{\mathbb{C}} \setminus f^{-1}(f(B))$ s.t $f(a_i)=c$ so we have $f: W_i \xrightarrow{\text{homeo}} V_i$ ($\ni c$)
if we choose them W_i disjoint $V = V_1 \cap \ldots \cap V_d$ is a fundamental neighbourhood of c ($f: W_i \cap f^{-1}(V) \xrightarrow{\text{homeo}} V$).

<u>Example</u> $f(z) = \frac{z}{z^3+2}$ rational of deg 3. $f'(z) = \frac{2(1-z^3)}{(z^3+2)^2}$
so the: there are branch points at the cubic roots of unity $1, \omega, \omega^2$, $f(1) = \frac{1}{3}$, $f(\omega) = \frac{\omega}{3}$, $f(\omega^2) = \frac{\omega^2}{3}$, these pts have order 1 (for instance the sol of $f(z) = \frac{1}{3}$ are $1, 1, -2$)
similarly $f(z) = \frac{\omega}{3}$ has sol $\omega, \omega, -2\omega$ and $f(z) = \frac{\omega^2}{3}$ has sol $\omega^2, \omega^2, -2\omega^2$
When $z = \infty$, $(f \circ J)(z) = \frac{z^2}{1+2z^3}$ has a double zero at $z=0$, i.e a branch pt at ∞. The poles of f are simple pts.

We continue with $\mathrm{Aut}(\hat{\mathbb{C}})$, see Jones-Singerman or Beardon [2] for good accounts of the topic

- Now, we consider $Aut(\hat{\mathbb{C}})$, that is meromorphic / rational functions which are bijective so

Mobius transf. $= Aut(\hat{\mathbb{C}}) = \{ T = \frac{az+b}{cz+d}, \ a,b,c,d \in \mathbb{C}, \ ad-bc \neq 0 \}$

Observe that T does not determine the coeff.

a,b,c,d and $\lambda a, \lambda b, \lambda c, \lambda d$, $\lambda \in \mathbb{C} \setminus \{0\}$ correspond to the same transf. T. In fact $Aut(\hat{\mathbb{C}})$ is the Galois gr of $\mathbb{C} \subseteq \mathbb{C}(z)$

Mobius transf form a group (of homeomorphisms of $\hat{\mathbb{C}}$)

$$Aut(\hat{\mathbb{C}}) \simeq PGL(2,\mathbb{C}) \simeq PSL(2,\mathbb{C})$$

Any Mobius transf is a product of finitely many of the following transformations (classify by type)

i) $R_\theta(z) = e^{i\theta} z$ $(\theta \in \mathbb{R})$ rotation of S^2 by an angle θ with vertical axis

ii) $J(z) = 1/z$, we have seen it is a rotation by angle π around x_1-axis

iii) $S_r(z) = rz$, $r \in \mathbb{R}, r > 0$ (fixes 0 and ∞) and acts on $\mathbb{C}$ as a similarity

iv) $T_t(z) = z+t$, $t \in \mathbb{C}$, fixes ∞ and acts on $\mathbb{C}$ as translation

We know that two elements of T_1, T_2 of $PSL(2,\mathbb{C})$ are conjugate iff $tr^2(T_1) = tr^2(T_2)$

- T is elliptic if $0 \leq tr^2(T) < 4$
- T parabolic if $tr^2(T) = 4$
- T hyperbolic if $tr^2(T) > 4$
- T loxodromic if $tr^2(T) < 0$ or $tr^2(T) \notin \mathbb{R}$

A Mobius transf T is called a rotation if $\Phi T \Phi^{-1}$ is a rotation of S^2

$$Rot(\hat{\mathbb{C}}) \simeq PSU(2,\mathbb{C}) \simeq SO(3,\mathbb{R})$$

Finite groups of $Aut(\hat{\mathbb{C}})$ (conjugate to gr of rotations) are cyclic, dihedral, ~~rotation gr~~ A_4, Σ_4, A_5