Chapter 1 Elliptic Functions and Tori

We consider periodic functions $f: \mathbb{C} \to \mathbb{C}\ (\hat{\mathbb{C}})$

$\omega \neq 0$ is a period of f if $f(z+\omega) = f(z)\ \forall z \in \mathbb{C}$

We say f period if there is a period $\omega \neq 0$

Th Let Ω_f be the set of periods of a function f defined on $\mathbb{C}$, then Ω_f is a subgroup of $(\mathbb{C}, +)$ [Control the requirements for group]

Th Let Ω_f be the set of periods of a non-constant meromorphic function $f: \mathbb{C} \to \hat{\mathbb{C}}$. Then Ω_f is a discrete subset of $\mathbb{C}$

Proof (Sketch) If ~~Ω_f~~ Ω_f not discrete, we have a contradiction with the fact that poles and zeros of non-constant meromorphic functions are isolated.

Th Let Ω be a discrete subgroup of $\mathbb{C}$. Then one of the following holds : i) $\Omega = \{0\}$

ii) $\Omega = \{n\omega_1 \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$, for some $\omega_1 \in \mathbb{C} \setminus \{0\}$

iii) $\Omega = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z}$, for some $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ linearly independent over $\mathbb{R}$

One calls a function with period set of type ii) simply periodic, and with period set of type iii) doubly periodic

The set of periods Ω_f of a non-constant meromorphic funct is a discrete topological group (e.g. a lattice). Recall that if Ω is a discrete subgroup of a top. gr. G, and K compact in G, then $\Omega \cap K$ is finite

We consider now lattices: $\Omega = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$ ω_1, ω_2 lin. indep. over $\mathbb{R}$

$\Omega = \Omega(\omega_1, \omega_2)$

Consider $\begin{aligned} \omega_2' &= a\omega_2 + b\omega_1 \\ \omega_1' &= c\omega_2 + d\omega_1 \end{aligned}$. $\{\omega_1', \omega_2'\}$ is a basis for $\Omega(\omega_1, \omega_2)$ iff $ad - bc = \pm 1$

Given a lattice we define $z_1, z_2 \in \mathbb{C}$ to be congruent mod Ω if $z_1 - z_2 \in \Omega$, i.e. z_1, z_2 belong to the same Ω-orbit

A closed, connected subset P of $\mathbb{C}$ is defined to be a fundamental region for Ω if

i) $\forall z \in \mathbb{C}$, P contains at least one pt in the same Ω-orbit of z

ii) no two pts in the interior of P are in the same Ω-orbit

(iii) area of $P \setminus \mathring{P} = 0$)

If P is a Euclidean polygon, with a finite number of sides, P is called a fundamental polygon

If P is a fundamental region, given $t \in \mathbb{C}$, $P + t = P'$ also fund reg.

Example $\Omega(1, i)$ has a fundamental polygon with vertices $0, 1, i, 1+i$

Example The set $D(\Omega) = \{ z \in \mathbb{C} / |z| \leq |z - \omega| \ \forall \omega \in \Omega \}$ is a fundamental region, called Dirichlet region

Area of fundamental regions P_1, P_2 for a lattice coincide.

It is know that given a lattice Ω, $\mathbb{C}/\Omega = T$ is a surface (Ω acts freely on $\mathbb{C}$) with universal covering $\pi : \mathbb{C} \longrightarrow T$

Elliptic functions

Def A meromorphic function $f : \mathbb{C} \to \bar{\mathbb{C}}$ is elliptic w.r.t a lattice Ω if it is doubly periodic with Ω its period-set.

Elliptic comes from certain integrals generalizing $\int \sqrt{\frac{a^2 - ex^2}{a^2 - x^2}}\, dx$ (arc length of a ellipse)

The inverses of ~~certain~~ elliptic integrals are doubly periodic

If f is elliptic respect to a lattice Ω, then we may consider f as a function on the torus $T = \mathbb{C}/\Omega$. If we consider solutions of $f(z)=c$ (isolated as f meromorphic), any fundamental polygon P will have a finite number of sol (that we may assume are interior pts). If there are $z=z_1,\dots,z_r$ sol in P with mult $k_1,\dots,k_r$ and $\underline{N=\sum k_i}$ then we can think that $f([z])=c$ has N solutions, counting mult. We define the order $ord(f)$ of the elliptic function f to be the ~~number~~ sum of ~~poles~~ multiplicities of the Ω-classes of poles of f.

$\underline{Th}$. f is constant iff $N=0$

$\underline{Proof}$ $\Rightarrow$ Clear. $\Leftarrow$ Assume f has no poles, so f analytic on $\mathbb{C}$ and also on P (compact) so $f(P)$ ($=f(\mathbb{C})$) bounded and so f constant.

$\underline{Th}$ The sum of the residues of f in P is 0

As consequence $\underline{\text{there is no}}$ elliptic function of order 1.

As for rational functions

$\underline{Th}$ If f has order $N>0$, then f takes each value $c\in\hat{\mathbb{C}}$ exactly N times.

$\underline{Th}$ Let f and g be elliptic functions (the same lattice) with poles at the same pts of $\mathbb{C}$, and with the same principal parts. Then $f(z)=g(z)+c$

$\underline{Th}$ Let f and g be elliptic functions respect to Ω, with zeros and poles of the same order at the same pts of $\mathbb{C}$. Then $f(z)=cg(z)$

$\underline{Th}$ Let the congruence classes of zeros and poles of an elliptic function f be $[a_1],\dots,[a_r]$ and $[b_1],\dots,[b_s]$ with multiplicities $k_1, \dots, k_r$ and $l_1, \dots, l_s$. Then

$$\sum k_i a_i \sim \sum l_j b_j \mod \Omega$$

<u>Proof</u> As the formula is mod Ω we can assume that $a_i, b_j \in P$ (as usual no zeros or poles on ∂P)

First $\sum k_i a_i - \sum l_j b_j = \frac{1}{2\pi i}\int_{\partial P} \frac{z f'}{f} dz$

if f has a zero of mult. k at a $f(z) = (z-a)^k g(z)$

$$\frac{z f'(z)}{f(z)} = \frac{z\left[k(z-a)^{k-1}g(z) + (z-a)^k g'(z)\right]}{(z-a)^k g(z)} =$$

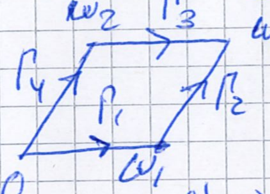
g analytic near a, $g(a) \neq 0$

$$= \frac{kz}{z-a} + \frac{z g'(z)}{g(z)}$$ near a

analytic

residue of $z\frac{f'}{f}$ at $z=a$ is ka. Similarly if f has a pole at $z=b$ of mult l, the residue of $z\frac{f'}{f}$ at $z=b$ is $-lb$. So we have proved

$$\sum k_i a_i - \sum l_j b_j = \frac{1}{2\pi i}\int_{\partial P} \frac{z f'}{f} dz$$



and we have as for rational functions

$$\int_{\Gamma_2} \frac{z f'(z)}{f(z)} = -\int_{\Gamma_4} \frac{z f'}{f(z)} + 2\pi n_1 i\omega_1 \; ; \int_{\Gamma_3} \frac{z f'(z)}{f} dz = -\int_{\Gamma_1} \frac{z f'}{f(z)} + 2\pi n_2 i\omega_2$$

So $\sum k_i a_i - \sum l_j b_j = \frac{1}{2\pi i}\int_{\partial P} \frac{z f'}{f} dz = \frac{1}{2\pi i}(2\pi i)(n_1\omega_1 + n_2\omega_2) \in \Omega$

To construct non-constant elliptic functions given a lattice $\Omega(\omega_1, \omega_2)$, we have that

<u>Th</u> If $s \in \mathbb{R}$, then $\sum'_{\omega \in \Omega} |\omega|^{-s}$ converges iff $s > 2$

As consequence for each integer $N \geq 3$

<u>Th</u> The function $F_N(z) = \sum_{\omega \in \Omega} (z-\omega)^{-N}$ is elliptic of order N with respect to Ω $(N \geq 3)$

For elliptic functions of order 2, we have Weierstraß pe-function

$$\wp(z) = \frac{1}{z^2} + \sum'_{\omega\in\Omega}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

by comparing $\wp(z)$ with $\sum' |\omega|^{-3}$ one sees that $\wp(z)$ is meromorphic, ~~with~~ all poles of order 2 at $\omega \in \Omega$

$\wp(z)$ is periodic with period set $\Omega_\wp$. To see it we calculate $\wp'(z) = -2\sum_\Omega (z-\omega)^{-3} = -2F_3(z)$, so $\wp'$ is elliptic respect to Ω (of order 3) so $\wp(z+\omega) - \wp(z) = c_\omega$ $\forall z$

If $z = -\omega/2$ $\wp(\omega/2) - \wp(-\omega/2) = 0$, as $\wp$ even

so $\wp(z+\omega) = \wp(z)$ $\forall z \in \mathbb{C}$, $\omega \in \Omega$, $\Omega \subseteq \Omega_\wp$. Now 0 is a pole of $\wp(z)$, $\forall \omega \in \Omega_\wp$ there is a pole (the only poles) so $\Omega_\wp \subseteq \Omega$ and $\Omega = \Omega_\wp$

Notice that there is only a class of poles, each of order 2 So $\wp(z)$ has order 2, In the same way $\wp'$ has a unique class of poles of order 3, $\mathrm{ord}(\wp') = 3$.

One has $\wp'(z)^2 = 4\wp(z)^3 - 60G_4\,\wp(z) - 140G_6 = p(\wp(z))$

with $G_4 = G_4(\Omega) = \sum'_\Omega \omega^{-4}$, we write $60G_4 = g_2$, $140G_6 = g_3$

By taking $z = \wp(t)$, we get $t = \int \frac{dz}{\sqrt{p(z)}}$

Weierstraß functions do depend on the class of lattices and given a cubic polynomial $p(z) = 4z^3 - c_2 z - c_3$ with distinct roots there exists a lattice Ω s.t $c_2 = g_2(\Omega)$, $c_3 = g_3(\Omega)$. So each such polynomial defines a field of elliptic functions.

First if Ω is a lattice with basis $\{\omega_1, \omega_2\}$ and if a fundamental parallelogram P contains $0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2$ and $\frac{1}{2}(\omega_1+\omega_2)$ in $\wp$, then $\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}(\omega_1+\omega_2)$ are the 0's of $\wp'$ in P.

We have seen that 0 is a pole of order 3, and the only one $\wp'$ has 3 zeros in P, as $\wp'$ is odd $\wp'(\frac{\omega_1}{2}) = -\wp'(-\frac{\omega_1}{2})$ and as $\frac{\omega_1}{2} - (-\frac{\omega_1}{2}) \in \Omega$; $\wp'(\frac{\omega_1}{2}) = 0$, equally for $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$

Denote $e_j = \wp(\frac{1}{2}\omega_j)$ $j=1,2,3$, $\omega_3 = \omega_1+\omega_2$.

For each $c \in \hat{\mathbb{C}} \setminus \{e_1, e_2, e_3, \infty\}$ the equation $\wp(z) = c$ has two simple solutions, for $c = e_1, e_2, e_3$ and ∞ a double sol.

The functions $f_j(z) = \wp(z) - e_j$ are elliptic of order 2, and f_j has double zeros on $[\frac{1}{2}\omega_j]$, so $f_j(\frac{1}{2}\omega_k) = e_k - e_j \neq 0$ $k \neq j$

<u>So the zeros of $p(z) = z^3 - g_2 z - g_3$ are all distinct</u>

The field of elliptic functions (respect Ω)

Given a lattice and corresponding $\wp$ and $\wp'$ we can form the field rational functions on $\wp$ $\mathbb{C}(\wp)$ and the field of rational functions $\mathbb{C}(\wp, \wp') = \mathbb{C}(\wp)[p]$, $p = z^3 - g_2 z - g_3$. On the other hand elliptic functions form a field: the field of elliptic functions (where $\mathbb{C}$ is the subfield of constant elliptic functions). We have also the field of even elliptic functions

$$\mathbb{C} \subseteq E_{even}(\Omega) \subseteq E(\Omega)$$

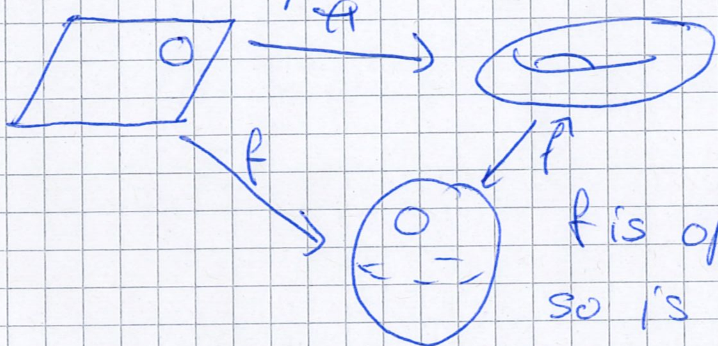
One sees that $E_{even}(\Omega) = \mathbb{C}(\wp)$ and $E(\Omega) = \mathbb{C}(\wp, \wp')$

<u>Th</u> If $f, g \in E(\Omega)$, then there exists a non-zero polynomial $\phi(x,y)$ (complex coeff) s.t $\phi(f(z), g(z)) \equiv 0$

Topology of elliptic functions

Given an elliptic function $f: \mathbb{C} \to \hat{\mathbb{C}}$ (respect to Ω) of $\mathrm{ord}(f) = N > 0$. f induces $\hat{f}: T = \mathbb{C}/\Omega \longrightarrow \hat{\mathbb{C}}$ $\hat{f}[z] = f(z)$

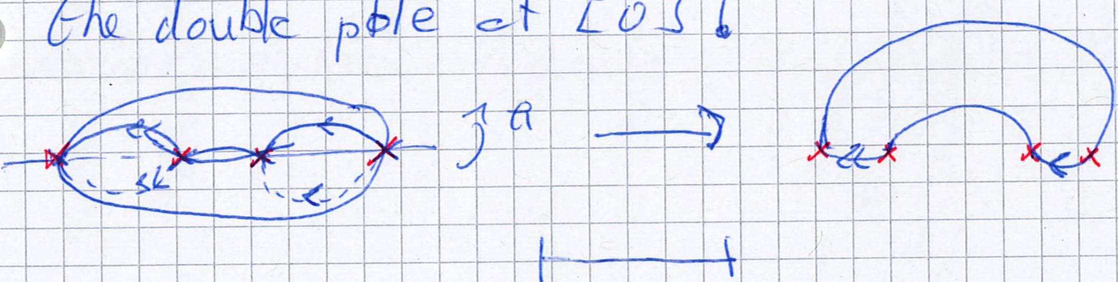
Take $a \in \mathbb{C}$ and $f(a) = c \in \hat{\mathbb{C}}$ with mult k. Take a neighbourhood U of a small enough, so that no two pts of U are congruent mod Ω $\quad \pi: \mathbb{C} \longrightarrow \mathbb{C}/\Omega$ takes $U \xrightarrow{\text{homeo}} \tilde{U} \ni [a]$



f is open and locally k-to-1, and so is $\hat{f}$

The branch pts of $\hat{f}$ are the zeros of $\hat{f}'$ and the multiple poles of $\hat{f}$, and this set is finite (f, f' meromorphic), again away from branch pts $\hat{f}$ is a covering map, the $\deg \hat{f} = N$

<u>Example</u> Weierstraß elliptic function $\wp$ is a branched covering $\hat{\wp}: T \longrightarrow \hat{\mathbb{C}}$ of degree 2 ramified at the zeros of $\hat{\wp}'$: $[\frac{1}{2}\omega_1]$, $[\frac{1}{2}\omega_2]$, $[\frac{1}{2}(\omega_1 + \omega_2)]$, and at the double pole at $[0]$!



We have seen that if Ω is a lattice of $\mathbb{C}$, $\mathbb{C}/\Omega = T$ is a surface. We see that the tori $\mathbb{C}/\Omega$ are Riemann surfaces, i.e they have analytic atlases (we use the fact that Ω is discrete subgroup of $\mathbb{C}$)

Consider $\delta = \inf_{\omega \in \Omega \setminus \{0\}} |\omega| > 0$ and let $\mathcal{V}$ be the set of all open discs in $\mathbb{C}$ of diameter at most $\delta/2$ then $\forall V \in \mathcal{V}, \omega \in \Omega \setminus \{0\}$ $V \cap V+\omega = \emptyset$ and given $V, V' \in \mathcal{V}$, V has non-empty intersection with at most one of $V'+\omega$, $\omega \in \Omega$. If we restrict $p: \mathbb{C} \to \mathbb{C}/\Omega$ to these discs V $p_V: V \to p(V) = U_V$ is homeomorphism. We have an atlas $\{(U_V, \varphi_V = p_V^{-1})\}$ for $\mathbb{C}/\Omega$. The only we have to see is that this atlas is analytic. Suppose that (U_V, φ_V) and $(U_{V'}, \varphi_{V'})$ are charts with $U_V \cap U_{V'} \neq \emptyset$, the coordinate function

$$\phi_{V'V} = \varphi_{V'} \varphi_V^{-1} : \varphi_V(U_V \cap U_{V'}) \to \varphi_{V'}(U_V \cap U_{V'})$$

~~each~~ $z \in \varphi_V(U_V \cap U_{V'})$ let $z' = \phi_{V'V}(z)$, then $p_{V'}(z') = \varphi_{V'}^{-1}\varphi_{V'}\varphi_V^{-1}(z) = p(z)$

thus $z - z' \in \Omega$, ~~and~~ $z - z' = \omega$, and $V \cap (V' + \omega) \neq \emptyset$, so by ω is constant for $z \in \varphi_V(U_V \cap U_{V'})$ but $z' = z - \omega$ is an analytic function.

Given a topological torus there are infinitely many analytic structures on it. But we have. The holomorphic transformations $f_{a,b}: \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ are the transformations $f_{\alpha,\beta}[z] = [\alpha z + \beta]$ $\alpha, \beta \in \mathbb{C}$ and $\alpha\Omega \subseteq \Omega'$; $f_{\alpha,\beta}$ is conformal homeomorphism iff $\alpha\Omega = \Omega'$, so $\mathbb{C}/\Omega$ and $\mathbb{C}/\Omega'$ conformal equiv. iff Ω and Ω' are similar. We know from complex analysis that if $\Omega(\omega_1, \omega_2)$ and $\Omega'(\omega_1', \omega_2')$, Ω and Ω' are similar iff

$$\omega_2' = \alpha(a\omega_2 + b\omega_1)$$
$$\omega_1' = \alpha(c\omega_2 + d\omega_1)$$

$a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$

Proof. Let $f: \mathbb{C}/\Omega \to \mathbb{C}/\Omega'$ if $f[z] = [z']$, there are open discs V and V' in $\mathbb{C}$ centred at z and z' mapped homeomorphically by p and p' onto neighborh. U and U' of $[z]$ and $[z']$.

$$\begin{tikzcd} V \arrow[r] \arrow[d, shift right, "p"'] & V' \arrow[d, shift left, "p'"] \\ U \arrow[u, shift right, "\varphi_V"'] \arrow[r, "f"', "\text{holomo}"] & U' \arrow[u, shift left, "\varphi_{V'}"] \\ {[z]} & {[z']'} \end{tikzcd}$$

$F = \varphi_{V'}\, f\, p : \varphi_V(U \cap f^{-1}(U')) \to \varphi_{V'}(U' \cap f(U))$

analytic

F is not unique determined by $[z]$, changing V' by any $V' + \omega'$, we replace F by $F + \omega$, but as $\mathbb{C}$ is simply connected one of these function elements extends to a single-valued function $\tilde{f}: \mathbb{C} \to \mathbb{C}$ s.t $f \circ p = p' \circ \tilde{f}$. $f[z] = [\tilde{f}(z)]'$ $\forall z \in \mathbb{C}$, for any fixed $\omega \in \Omega$

$\tilde{f}(z+\omega) = \tilde{f}(z) + \omega'_z$ (may be depending on z) now

$z \mapsto \omega'_z = \tilde{f}(z+\omega) - \tilde{f}(z)$ is continuous and Ω' dis. so ω'_z is constant $\forall z \in \mathbb{C}$, so $\tilde{f}(z+\omega) = \tilde{f}(z) + \omega'$ $\forall z \in \mathbb{C}$

$d\tilde{f}/dz$ is elliptic with respect to Ω, and analytic, so constant

$\tilde{f}(z) = az + b$, $f[z] = [az+b]'$, $a, b \in \mathbb{C}$

$\forall \omega \in \Omega$, $f[z+\omega] = f[z]$ $[a(z+\omega)+b]' = [az+b]'$ $\underline{a\omega \in \Omega'}$

Conversely if $a\Omega \subseteq \Omega'$ f is holomorphic

If f is homeomorphism, then its holomorphic inverse must have the form $[z]' \to [(z-b)/a]$; $a^{-1}\Omega' \subseteq \Omega$

$\Omega' \subseteq a\Omega$.

$\mathrm{Aut}(\mathbb{C}/\Omega) = \{ f_{\alpha,\beta} \mid f_{\alpha,\beta}[z] = [\alpha z + \beta], \alpha, \beta \in \mathbb{C}, \alpha\Omega = \Omega \}$