TORELLI'S PROBLEM FOR TWO-PUNCTURED TORI

ENRIQUE ARTAL BARTOLO

Let T be a complex torus of the form $\mathbb{C}/\Lambda$; for simplicity we assume that Λ is the lattice generated by $1, \tau \in \mathbb{C}$, $\Im\tau > 0$. We want to study this torus with two punctures; we may assume that one of them corresponds to $P := \Lambda = [0] \in T$ while the other one is $Q := [\rho] \in T$, for some $\rho \in \mathbb{C} \setminus \Lambda$. We will denote $X := T \setminus \{P, Q\}$.

In order to study it we want to consider the generalized Weierstraß functions which are meromorphic functions $\mathbb{C} \dashrightarrow \mathbb{C}$ depending on the lattice Λ and some $\rho \in \mathbb{C}$:

$$\wp_{\Lambda,\rho}(z) := \sum_{\lambda\in\Lambda\setminus\{0,\rho\}} \left(\frac{1}{(z-\lambda)(z-\lambda+\rho)} - \frac{1}{\lambda(\lambda-\rho)}\right) + \sum_{\lambda\in\Lambda\cap\{0,\rho\}} \frac{1}{(z-\lambda)(z-\lambda+\rho)}.$$

These series converge absolutely and uniformly on compacts and hence they define meromorphic functions. Note that for $\rho = 0$ we recover the classical Weierstraß function. Their derivatives are

$$\wp'_{\Lambda,\rho}(z) = \frac{1}{\rho}\sum_{\lambda\in\Lambda}\left(\frac{1}{(z-\lambda+\rho)^2} - \frac{1}{(z-\lambda)^2}\right),$$

which are clearly Λ-invariants. Hence,

$$\forall\lambda\in\Lambda, \exists\tilde{\mu}_\lambda(\rho)\in\mathbb{C} \text{ such that } \wp_{\Lambda,\rho}(z+\lambda) = \wp_{\Lambda,\rho}(z) + \tilde{\mu}_\lambda(\rho).$$

Properties 0.1. Let us state some immediate properties (and well-known at least for the classical $\wp_\Lambda := \wp_{\Lambda,0}$).

1. $\wp_{\Lambda,0}$ is even and hence Λ-periodic, i.e., $\tilde{\mu}_\lambda(0) = 0$, $\forall\lambda\in\Lambda$.
2. $\tilde{\mu}_{\lambda_1+\lambda_2} = \tilde{\mu}_{\lambda_1} + \tilde{\mu}_{\lambda_2}$.
3. $\wp_{\Lambda,\lambda}$ is holomorphic $\forall\lambda\in\Lambda\setminus\{0\}$.

Let us compute the derivative of μ_λ with respect to ρ:

$$\mu'_{\lambda_0}(\rho) = \sum_{\lambda\in\Lambda}\left(\frac{1}{(z-\lambda)(z-\lambda+\rho)^2} - \frac{1}{(z+\lambda_0-\lambda)(z+\lambda_0-\lambda+\rho)^2}\right) = 0.$$

Hence, we obtain that $\wp_{\Lambda,\rho}$ is always Λ-periodic; in particular:

$$\wp_{\Lambda,\lambda}(z) = -\frac{2}{\lambda^2}.$$

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Remark 0.1. Let us recall that a primitive of $\wp_\Lambda$ is the opposite of the Weierstraß zeta function

$$\zeta_\Lambda(z) := \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{z-\lambda}+\frac{z}{\lambda^2}+\frac{1}{\lambda}\right)+\frac{1}{z}.$$

Since its derivative is also periodic, we have that

$$\mu(\lambda) := \zeta_\Lambda(z+\lambda)-\zeta_\Lambda(z),\quad \lambda\in\Lambda,$$

does not depend on z and it is additive: $\mu(\lambda_1+\lambda_2)=\mu(\lambda_1)+\mu(\lambda_2)$. It is easily seen that

$$\mu(1)=3-\sum_{\lambda\in\Lambda\setminus\{0,1\}}\frac{1}{\lambda^2(\lambda-1)}$$

Let us study now $\operatorname{Alb}(X)=(H^0(T;\Omega^1_T\log(P+Q)))^*/H_1(X;\mathbb{Z})$. We consider the following two basis generators for $H^0(T;\Omega^1\log(P+Q))$. One is the holomorphic 1-form on T defined by the Λ-invariant holomorphic 1-form $\omega := dz$. The second one (to be changed later) is the log-meromorphic 1-form on T defined by the Λ-invariant meromorphic 1-form

$$\eta_0 := \frac{-\rho}{2i\pi}\wp_{\Lambda,\rho}(z)dz.$$

For $H_1(X;\mathbb{Z})$, we consider a basis of three cycles. Fix a generic $\sigma\in\mathbb{C}$; we ask to be outside the real lines generated $\mathbb{R}$, $\tau\mathbb{R}$, $\rho+\mathbb{R}$ and $\rho+\tau\mathbb{R}$ and their translated by Λ. Then, we consider α to be defined by the segment $[\sigma,\sigma+1]$, β to be defined by the segment $[\sigma,\sigma+\tau]$ and γ as a *small* meridian around $-\rho$.

Remark 0.2. Note that the homology classes of α and β are not changed by small perturbations of σ, but they depend on σ (by adding suitable multiples of γ).

The following integrals are easily computed

$$\int_\alpha\omega=1,\quad \int_\beta\omega=\tau,\quad \int_\gamma\omega=0,\quad \int_\gamma\eta_0=1.$$

Let $A(\rho):=\int_\alpha\eta_0$ and $\eta=\eta_0-A\omega$. Note that

$$\int_\gamma\eta=1,\quad \int_\alpha\eta=1,\quad\text{and}\quad \int_\beta\eta=B,$$

to be determined. We start by computing $A(\rho)$ (we assume $\rho\notin\Lambda$). Note that

$$\frac{-\rho}{2i\pi}\wp_{\Lambda,\rho}(z)=\frac{1}{2i\pi}\sum_{\lambda\in\Lambda\setminus\{0\}}\left(\frac{1}{z-\lambda+\rho}-\frac{1}{z-\lambda}+\frac{\rho}{\lambda(\lambda-\rho)}\right)+\frac{1}{2i\pi}\left(\frac{1}{z+\rho}-\frac{1}{z}\right).$$

Hence

$$A(\rho)=\frac{1}{2i\pi}\sum_{\lambda\in\Lambda\setminus\{0\}}\left(\log\frac{z-\lambda+\rho}{z-\lambda}\bigg|_\sigma^{\sigma+1}+\frac{\rho}{\lambda(\lambda-\rho)}\right)+\frac{1}{2i\pi}\log\frac{z+\rho}{z}\bigg|_\sigma^{\sigma+1}.$$

For later use we need to compute

$$\lim_{\rho\to 0}\frac{A(\rho)}{\rho}=\lim_{\rho\to 0}A'(\rho)=A'(0)$$

which equals

$$\frac{1}{2i\pi}\left(\sum_{\lambda\in\Lambda\setminus\{0\}}\left(\frac{-1}{(\sigma-\lambda)(\sigma+1-\lambda)}+\frac{1}{\lambda^2}\right)-\frac{1}{\sigma(\sigma+1)}\right),$$

i.e.,

$$-\frac{1}{2i\pi}\left(\wp_{\Lambda,1}(\sigma)-1+\sum_{\lambda\in\Lambda\setminus\{0,1\}}\frac{1}{\lambda^2(\lambda-1)}\right)=\frac{\mu(1)}{2i\pi}.$$

There is some ambiguity on the choice of the determination of the logarithm but we may consider that $A(\rho)$ is *well-defined* $\mathrm{mod}\mathbb{Z}$. The function defining η is:

$$\frac{1}{2i\pi}\sum_{\lambda\in\Lambda}\left(\frac{1}{z-\lambda+\rho}-\frac{1}{z-\lambda}-\log\frac{\sigma+1-\lambda+\rho}{\sigma+1-\lambda}+\log\frac{\sigma-\lambda+\rho}{\sigma-\lambda}\right).$$

Hence

$$B=\frac{1}{2i\pi}\sum_{\lambda\in\Lambda}\left(\log\frac{\sigma+\tau-\lambda+\rho}{\sigma+\tau-\lambda}-\tau\log\frac{\sigma+1-\lambda+\rho}{\sigma+1-\lambda}+(\tau-1)\log\frac{\sigma-\lambda+\rho}{\sigma-\lambda}\right).$$

As before, this number is also well-defined $\mathrm{mod}\Lambda$. This value depends on ρ; it defines a (multi-valued) function whose derivative is

$$B'(\rho)=\frac{1}{2i\pi}\sum_{\lambda\in\Lambda}\left(\frac{1}{\sigma+\tau-\lambda+\rho}-\tau\frac{1}{\sigma+1-\lambda+\rho}+(\tau-1)\frac{1}{\sigma-\lambda+\rho}\right)$$

and should not depend on σ, so it does not depend on ρ, and hence it is constant. Hence $B(\rho)=b\rho$ for some constant ρ (note that since $\lim_{\rho\to 0}\eta=0$, then $B(0)=0$).

If $\rho\to 0$, in some natural sense $\wp_{\Lambda,\rho}\to\wp_\Lambda$, and $A(\rho)\to 0$, hence

$$\frac{\eta_0}{\rho}\to\frac{-\wp_\Lambda(z)\omega}{2i\pi},\qquad \frac{\eta}{\rho}=\frac{\eta_0-A(\rho)\omega}{\rho}\to-\frac{\wp_\Lambda(z)+\mu(1)}{2i\pi}\omega$$

Hence

$$b=\lim_{\rho\to 0}\int_\beta\frac{\eta}{\rho}=\frac{-1}{2i\pi}\int_\beta(\wp_\Lambda(z)+\mu(1))\,\omega=\frac{\zeta_\Lambda(\sigma+\tau)-\zeta_\Lambda(\sigma)-\tau\mu(1)}{2i\pi}=\frac{\mu(\tau)-\tau\mu(1)}{2i\pi}=1,$$

by Legendre's identity.

Theorem 0.3. $\mathrm{Alb}(X)\cong\mathbb{C}^2/\Gamma$ *where* Γ *is the lattice generated by the columns of the matrix*

$$\begin{pmatrix}1&0&\tau\\0&1&\rho\end{pmatrix}$$

It is the period matrix

$$\left(\int_{\alpha_j}\omega_i\right),$$

where $\omega_1=\omega$*,* $\omega_2=\eta$*,* $\alpha_1=\alpha$*,* $\alpha_2=\gamma$ *and* $\alpha_3=\beta$*.*

Departamento de Matemáticas, IUMA, Universidad de Zaragoza, c/ Pedro Cerbuna, 12, E-50009 Zaragoza SPAIN
E-mail address: `artal@unizar.es`