Linköpings tekniska högskola
Matematiska institutionen

Kurskod: TAIU08
Provkod: TEN1

Tentamen i TAIU08 (Flervariabelanalys)
2025-08-25 kl 14-19

Inga hjälpmedel tillåtna. Uppgifterna bedöms med 0-3 poäng.
Betyggränser: 8/11/14 poäng ger betyg 3/4/5.

1. (i) Låt $f(x, y) = y^2\sqrt{x}$. Beräkna $f'_y(9, 1)$ genom att använda definitionen av partiell derivata som ett gränsvärde. (1p)

   (ii) Bestäm alla lösningar $f(x, y)$ till den partiella differentialekvationen
   $$xf'_x - 2yf'_y = 18y$$
   genom att byta till variablerna $u = x^2y$ och $v = y$. (2p)

2. (i) Bestäm punkten på kurvan $(x(t), y(t), z(t)) = (e^t, \cos^2 t, \sin t)$, $t \in R$, som svarar mot $t = \frac{\pi}{4}$ samt en tangentvektor till kurvan i punkten (2p)

   (ii) Beräkna riktningsderivatan av $f(x, y, z) = xyz^4$ i punkten $(-1, 1, 2)$ i den riktning som ges av vektorn $(1, -1, 1)$ (1p).

3. Bestäm alla lokala maximi- och minimipunkter samt sadelpunkter till funktionen $f(x, y) = x^3 + 3xy^2 - 15x - 12y + 10$.

4. Bestäm största och minsta värde av $f(x, y) = xy^2 - x + 3y + 1$ på det område D som ges av olikheterna $0 \leq y \leq -x \leq 2$.

5. Beräkna dubbelintegralen $\int\int_D (x - y) \cdot e^{x+y}\, dxdy$,
   där D är triangeln med hörn i $(0, 0), (1, 1), (0, 2)$.

6. Beräkna trippelintegralen $\int\int\int_D z \cdot \sqrt{x^2 + y^2 + z^2} dxdydz$,
   där $D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 2, x \geq 0, y \leq 0, z \leq 0\}$

1 (i) $f(x,y) = y^2\sqrt{x}$, $f'_y(9,1)$

$$f'_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} ;$$

$$f'_y(9,1) = \lim_{k\to 0} \frac{f(9,1+k) - f(9,1)}{k} = \lim_{k\to 0} \frac{3(1+k)^2 - 3\cdot 1}{k} =$$

$$= \lim_{k\to 0} \frac{6k + 3k^2}{k} = \lim_{k\to 0}(6 + 3k) = \underline{6}$$

(ii) $x f'_x - 2y f'_y = 18y$ lösningsmängd = ?

Variabelbyte $u = x^2 y$, $v = y$ | $u'_x = 2xy$, $u'_y = x^2$, $v'_x = 0$, $v'_y = 1$

$$f'_x = f'_u \cdot u'_x + f'_v \cdot v'_x = f'_u \cdot 2xy + f'_v \cdot 0 = \underline{2xy \cdot f'_u}$$

$$f'_y = f'_u \cdot u'_y + f'_v \cdot v'_y = f'_u \cdot x^2 + f'_v \cdot 1 = \underline{x^2 f'_u + f'_v}$$

Insättning: $x \cdot 2xy f'_y - 2y(x^2 f'_u + f'_v) = 18y$

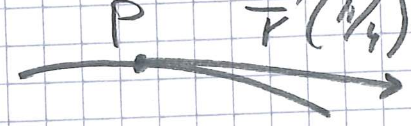
$-2y f'_v = 18y$ eller $\underline{f'_v = -9}$

$\Rightarrow$ Integrera på v: $f = \int f'_v \, dv = \int -9 \, dv = -9v + c(u)$

$\Rightarrow$ $f(x,y) = -9\cdot y + c(x^2 \cdot y)$, där $c(\cdot)$ är en godtycklig C' funktion.

2 (i) $\bar{r}(t) = (e^t, \cos^2 t, \sin t)$, $P = ?$ ~ $t = \pi/4$

$$P(e^{\pi/4}, \cos^2\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}) = \underline{\left(e^{\pi/4}, \overbrace{\left(\tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{2}}\right)}^{=1/2}, \tfrac{1}{\sqrt{2}}\right)}$$



$\bar{r}'(\pi/4) = ?$

$$\bar{r}'(t) = (e^t, 2\cos t\cdot(-\sin t), \cos t)$$

$$\bar{r}'\left(\tfrac{\pi}{4}\right) = \left(e^{\pi/4}, 2\cdot\tfrac{1}{\sqrt{2}}\cdot\left(-\tfrac{1}{\sqrt{2}}\right)\cdot \tfrac{1}{\sqrt{2}}\right) =$$

$$= \underline{\left(e^{\pi/4}, -1, \tfrac{1}{\sqrt{2}}\right)}$$

(ii) $f(x,y,z) = xyz^4$, $P(-1,1,2)$, $\bar{v} = (1,-1,1)$

$|\bar{v}| = \sqrt{3}$

$$f'_{\frac{\bar{v}}{|\bar{v}|}}(P) = \nabla f(P)\cdot\frac{\bar{v}}{|\bar{v}|} = ?$$

$$\nabla f = (f'_x, f'_y, f'_z) = (yz^4, xz^4, 4xyz^3)$$

$$\nabla f(P) = (1\cdot 2^4, (-1)\cdot 2^4, 4\cdot(-1)\cdot 1\cdot 2^3) = (16, -16, -32)$$

$$\Rightarrow f'_{\frac{\bar{v}}{|\bar{v}|}}(P) = (16,-16,-32)\cdot\frac{1}{\sqrt{3}}(1,-1,1) = \frac{1}{\sqrt{3}}(16+16-32) = \underline{0}$$

3 $f(x,y) = x^3 + 3xy^2 - 15x - 12y + 10$ <u>Lok. undersök</u>

(i) stationära punkter: $\nabla f = \bar{0}$

$$\begin{cases} 3x^2 + 3y^2 - 15 = 0 \\ 6xy - 12 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 = 5 \\ xy = 2 \end{cases} \qquad y = \frac{2}{x}$$

$x^2 + \left(\frac{2}{x}\right)^2 = 5$ eller $x^4 + 4 = 5x^2$

$x^2 = t \geq 0$, $t^2 - 5t + 4 = 0$ $t_{12} = \underline{1, 4}$

$t_1 = 1$: $x = \pm 1 \Rightarrow P_1(1,2)$, $P_2(-1,-2)$

$t_2 = 4$: $x = \pm 2 \Rightarrow \underline{P_3(2,1), P_4(-2,-1)}$

(ii) P: $Q(h,k) = f''_{xx}(P)h^2 + 2f''_{xy}(P)hk + f''_{yy}(P)k^2$

$f''_{xx} = (f'_x)'_x = \underline{6x}$, $f''_{xy} = (f'_x)'_y = \underline{6y}$, $f''_{yy} = \underline{6x}$

	P_1	P_2	P_3	P_4
$f''_{xx}(P)$	6	−6	12	−12
$f''_{xy}(P)$	12	−12	6	−6
$f''_{yy}(P)$	6	−6	12	−12

P_1: $Q_1(h,k) = 6h^2 + 24hk + 6k^2 = 6(h^2 + 4hk + k^2) =$

$= 6((h+2k)^2 - 3k^2)$, indefinit, $\underline{\text{sadelpunkt.}}$

P_2: $\underline{\text{sadelpunkt.}}$

P_3: $Q_3(h,k) = 12h^2 + 12hk + 12k^2 = 12(h^2 + hk + k^2) =$

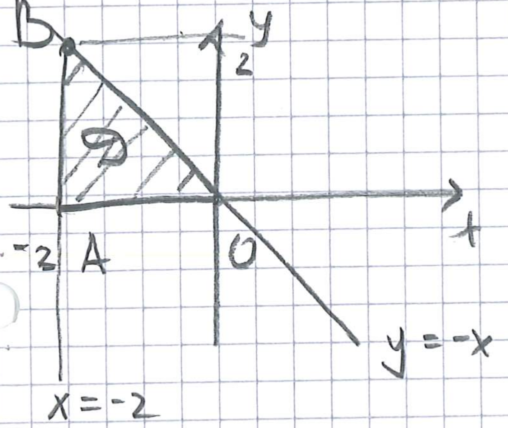
$= 12\left(\left(h + \frac{k}{2}\right)^2 + \frac{3}{4}k^2\right)$, pos. def., $\underline{\text{str. lok. minimipunkt}}$

P_4: $\underline{\text{str. lok. maximipunkt}}$

4) $f(x,y) = xy^2 - x + 3y + 1$

$D = \{ 0 \le y \le -x \le 2 \}$.

$y=0,\ y=-x,\ -x=2$ eller $y=0,\ y=-x,\ x=-2$



Obs 1) D är kompakt
2) f är kontinuerlig
$\Rightarrow$ max, min av f finns!

(i) Stationära (inre) punkter:

$$\nabla f = \bar{0} \iff \begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases} \iff \begin{cases} y^2 - 1 = 0 \\ 2xy + 3 = 0 \end{cases} \qquad y = \pm 1$$

$\Rightarrow y_1 = 1,\ x_1 = -3/2$ $\quad P_1(-3/2, 1)$ är kandidatpunkt.

$y_2 = -1,\ x_2 = 3/2$ $\quad P_2$ ligger utanför

$$f(P_1) = \underbrace{-\frac{3}{2}\cdot 1^2 - \left(-\frac{3}{2}\right)}_{=0} + 3\cdot 1 + 1 = \underline{\underline{4}} \quad \text{(kandidatvärde)}$$

(ii) Randen $= OA \cup AB \cup OB$

OA: $\begin{cases} x = t \\ y = 0 \end{cases}$ $\quad f\big|_{OA} = f(t,0) = 0 - t + 0 + 1 = 1 - t = g(t),$ $\quad t \in [-2, 0]$

$g' = 0,\ \emptyset,$ $\quad g(-2) = f(A) = 1 - (-2) = \underline{3}$
$g(0) = \underline{1} = f(O)$ $\quad$ Kandidatvärde

(5)

AB: $\begin{cases} x=-2 \\ y=t, \quad t\in[0,2] \end{cases}$ $f\Big|_{AB} = f(-2,t) = -2\cdot t^2 - (-2) + 3t + 1 = -2t^2 + 3t + 3 = h(t),\ t\in[0,2]$

$h'(t)=0 \iff -4t+3=0 \iff t=\frac{3}{4}\in(0,2) \Rightarrow P_3\left(-2,\frac{3}{4}\right)$

$h\left(\frac{3}{4}\right) = f(P_3) = -2\left(\frac{3}{4}\right)^2 + 3\cdot\frac{3}{4} + 3 = \frac{9}{8}+3 = \frac{33}{8}$

Kandidat värde

$f(B) = h(2) = -8+6+3 = \underline{1}$

OB: $\begin{cases} x=t \\ y=-t, \quad t\in[-2,0] \end{cases}$ $f\Big|_{OB} = f(t,-t) = t^3 - t - 3t + 1 = t^3 - 4t + 1 = r(t),\ t\in(-2,0)$

$r'(t)=0 \iff 3t^2-4=0 \iff t=\pm\frac{2}{\sqrt{3}}$ Obs $t=-\frac{2}{\sqrt{3}}\in(-2,0)$!

$\Rightarrow r\left(-\frac{2}{\sqrt{3}}\right) = -\frac{8}{3\sqrt{3}} - 4\cdot\left(-\frac{2}{\sqrt{3}}\right) + 1 = \frac{2}{3}\cdot\frac{8}{\sqrt{3}} + 1 = \underline{\frac{16}{3\sqrt{3}}+1}$

Kandidat värden: $4,\ 3,\ 1,\ \frac{33}{8},\ \frac{16}{3\sqrt{3}}+1$

$\Rightarrow \min f = 1$ antas i $(0,0)$ o $(-2,2)$

$\max f = \frac{33}{8}$ antas i $\left(-2,\frac{3}{4}\right)$

ty $\frac{33}{8} > \frac{16}{3\sqrt{3}}+1 \iff \frac{25}{8} > \frac{16}{3\sqrt{3}} \iff$

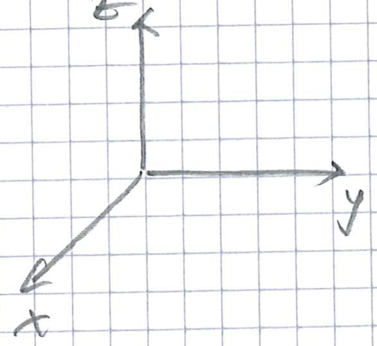
$\frac{625}{64} > \frac{256}{27}$ Obs $\frac{625}{64} > \frac{256}{27}$

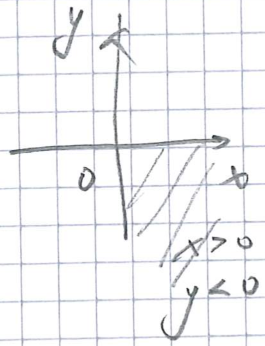
(6) $I = \iiint_{D} z\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$

$D = \{ x^2+y^2+z^2 \le 2,\ x \ge 0,\ y \le 0,\ z \le 0 \}$

$\pi/2 \le \theta \le \pi$

sfärisk





$\Rightarrow \frac{3\pi}{2} \le \varphi \le 2\pi$

$0 \le r \le \sqrt{2}$

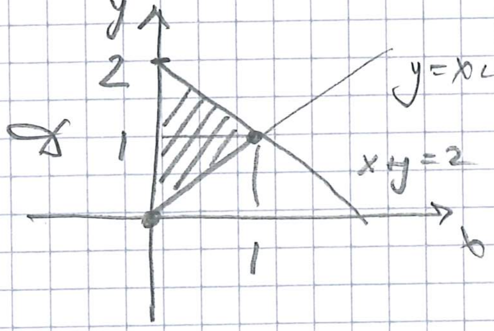
$$\Rightarrow I = \int_0^{\sqrt{2}} \left(\int_{\frac{\pi}{2}}^{\pi} \left(\int_{\frac{3\pi}{2}}^{2\pi} r\cos\theta \cdot r \cdot r^2 \sin\theta \, d\varphi \right) d\theta \right) dr =$$

$$= \frac{\pi}{2} \cdot \int_0^{\sqrt{2}} r^4\,dr \cdot \int_{\pi/2}^{\pi} \cos\theta \sin\theta \, d\theta$$

$$\frac{r^5}{5}\Big|_0^{\sqrt{2}} = \frac{4}{5}\sqrt{2} \qquad \frac{\sin^2\theta}{2}\Big|_{\pi/2}^{\pi} = \frac{1}{2}(0-1) = -\frac{1}{2}$$

$$= \frac{\pi}{2} \cdot \frac{4}{5}\sqrt{2} \cdot \left(-\frac{1}{2}\right) = \underline{-\frac{\pi\sqrt{2}}{5}}$$

⑤ $I = \iint_{D} (x-y)\, e^{x+y} \, dx dy$, D är triangel med hörn $(0,0), (1,1), (0,2)$ ⑥

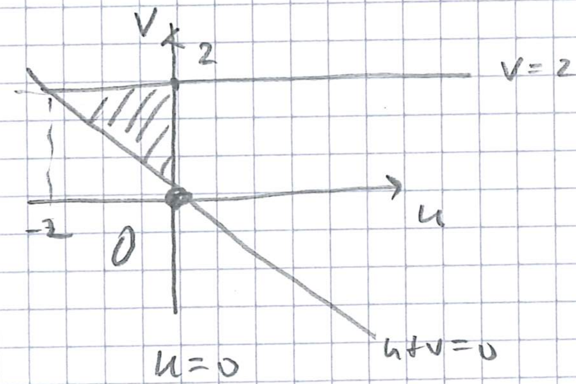


$x=0$
$(u+v=0)$

$$\begin{cases} u = x-y \\ v = x+y \end{cases} \oplus \Rightarrow u+v = 2x, \quad x = \frac{u+v}{2}$$

$$\frac{d(x,y)}{d(u,v)} = \frac{1}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{1}{2}$$

D_{uv}: $u=0$, $v=2$, $u+v=0$



$$I = \int_{-2}^{0} \left(\int_{-u}^{2} u e^{v} \, dv \right) du =$$

$$= \int_{-2}^{0} u \, e^{v} \Big|_{-u}^{2} du = \int_{-2}^{0} u \left(e^{2} - e^{-u} \right) du$$

$$= \underbrace{\int_{-2}^{0} u e^{2} du}_{I_1} - \underbrace{\int_{-2}^{0} u e^{-u} du}_{I_2} = -2e^{2} + 1 + e^{2} = \underline{1 - e^{2}}$$

$$I_1 = e^{2} \cdot \frac{u^{2}}{2} \Big|_{-2}^{0} = 0 - \frac{e^{2} \cdot 4}{2} = -2e^{2}$$

$$I_2 = \text{p.i.} = \left[-u e^{-u} - e^{-u} \right]_{-2}^{0} = (0-1) - (2e^{2} - e^{2}) = -1 - e^{2}$$