Linköpings universitet
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Tentamen i Analys i en variabel del 2, utbildningskod TAIU10, modul TEN2. 2025-08-22 , kl 14.00 – 19.00

Penna,radergummi,linjal,och passare får användas. Formelsamlingar och andra hjälpmedel är ej tillåtna. Lösningarna skall vara fullständiga,välmotiverade,ordentligt skrivna och avslutade med ett svar. Svaren ska förstås ges på så enkel form som möjligt.

Uppgifterna bedöms med 0 – 3 poäng. För betyg n (n = 3, 4 eller 5) krävs minst 4(n-1) poäng. Godkänd dugga 3 ger 1-2 bonuspoäng. Observera att bonus enbart gäller för betyget 3.

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1) Beräkna

a) $\displaystyle\int_1^2 \frac{2}{x^2+2x}\,dx$ b) $\displaystyle\int e^{\sin x}\cos x\,dx$ c) $\displaystyle\int x^5\cos x^3\,dx$.

2) Bestäm den lösning till differentialekvationen

$$y' = y\frac{x}{3+x^2}$$

som uppfyller villkoret $y(1) = 4$.

3) Området mellan kurvan $y = \sin\frac{x}{2}$, $0 \le x \le \pi$, linjen $x = \pi$ och x-axeln roteras ett varv kring y-axeln. Bestäm den rotationsvolym som uppkommer.

4) Beräkna

a) $\displaystyle\lim_{x\to 0}\frac{\sin(2x)-2x}{x(1-\cos x)}$ b) $\displaystyle\lim_{x\to 1}\frac{(x^{\frac{1}{3}}-1)(x-1)}{\arctan(2x-2)^2}$ c) $\displaystyle\int_0^\infty \frac{dx}{4+9x^2}$.

5) Bestäm alla lösningar till differentialekvationen

$$y''(x)-3y'(x)+2y(x)=(2x+5)e^{3x}$$

6) Bestäm alla deriverbara funktioner y som uppfyller integralekvationen

$$y(x)+\int_0^x \frac{y(t)}{1+t^2}\,dt = 2\arctan x\ .$$

7) En kurva ges i polära koordinater av $r = \frac{3}{7}\varphi^2$, $0 \le \varphi \le a$. Bestäm konstanten a så att kurvans längd blir 8.

Kortfattade lösningsförslag

1 a.
$$\int_1^2 \frac{2}{x(x+2)}\,dx \underset{PBU}{=} \int_1^2 \left(\frac{A=1}{x}+\frac{B=-1}{x+2}\right)dx =$$

$$= \int_1^2 \left(\frac{1}{x}-\frac{1}{x+2}\right)dx = \Big[\ln|x| - \ln|x+2|\Big]_1^2 =$$

$$= \ln 2 - \ln 4 - (\underbrace{\ln 1}_{=0} - \ln 3) = \ln 3 - \ln 2 = \ln\frac{3}{2}$$

b
$$\int e^{\sin x}\cos x\,dx = \left/\begin{matrix} t=\sin x \\ dt = \cos x\,dx\end{matrix}\right/ =$$

$$= \int e^t\,dt = e^t + C = e^{\sin x} + C$$

c.
$$\int x^5 \cos x^3\,dx = \frac{1}{3}\int 3x^2 \cdot x^3 \cdot \cos x^3\,dx =$$

$$= \left/\begin{matrix} t = x^3 \\ dt = 3x^2\,dx\end{matrix}\right/ = \frac{1}{3}\int \underset{\downarrow}{t}\,\underset{\uparrow}{\cos t}\,dt \underset{p.i}{=}$$

$$= \frac{1}{3}\left(t\sin t - \int \sin t\,dt\right) = \frac{1}{3}t\sin t + \frac{1}{3}\cos t + C$$

$$= \frac{1}{3}\left(x^3\sin x^3 + \cos x^3\right) + C$$

2.
$$y' = y\,\frac{x}{3+x^2}$$

$$\frac{dy}{dx} = y\,\frac{x}{3+x^2}, \quad y \neq 0.$$

$$\int \frac{1}{y}\,dy = \int \frac{x}{3+x^2}\,dx$$

$$\ln|y| = \frac{1}{2}\ln(3+x^2) + C$$

$y(1)=4$ ger $\ln 4 = \frac{1}{2}\ln(3+1^2) + C$

$$C = \frac{1}{2}\ln 4$$

$$\ln|y| = \frac{1}{2}\ln(3+x^2) + \frac{1}{2}\ln 4$$

$$\ln|y| = \frac{1}{2}\ln(12+4x^2)$$

$$|y| = \sqrt{12+4x^2}$$

$$y = \pm 2\sqrt{3+x^2}$$

$y(1)=4$ ger $y = 2\sqrt{3+x^2}$

Alt. lösning. Se nästa sida.

$$y' - \frac{x}{3+x^2}\, y = 0$$

$$\left(i.f = e^{\int -\frac{x}{3+x^2}dx} = e^{-\frac{1}{2}\ln(3+x^2)} = e^{\ln(3+x^2)^{-1/2}} = \frac{1}{\sqrt{3+x^2}} \right)$$

mult. med i.f ger

$$y' \frac{1}{\sqrt{3+x^2}} - \frac{x}{(3+x^2)^{3/2}}\, y = 0$$

$$\left(y \frac{1}{\sqrt{3+x^2}} \right)' = 0$$

$$y \frac{1}{\sqrt{3+x^2}} = C$$

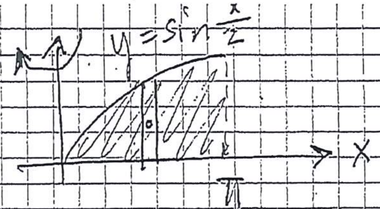
$$y = C\sqrt{3+x^2}$$

$y(1) = 4$ ger $4 = C\sqrt{3+1^2}$

$$C = 2$$

Svar: $y = 2\sqrt{3+x^2}$

3.



$$V = \int (\text{TP:s väg})\ \text{arean} =$$

$$= \int_0^{\pi} 2\pi x y\, dx = 2\pi \int_0^{\pi} x \sin\frac{x}{2}\, dx = \text{(P.I.)}$$

$$= 2\pi \left(\left[-\frac{x\cos\frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi} + \int_0^{\pi} 2\cos\frac{x}{2}\, dx \right) =$$

$$= 2\pi \left(0 + \left[\frac{2\sin\frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi} \right) =$$

$$= 8\pi \left(\sin\frac{\pi}{2} - 0 \right) = 8\pi \text{ v.e.}$$

Svar: 8π v.e.

4 a. $\lim\limits_{x\to 0} \dfrac{\sin 2x - 2x}{x(1-\cos x)} =$

$= \left/ \begin{array}{l} \text{Maclaurinutv.} \\ \sin t = t - \dfrac{t^3}{3!} + O(t^5) \\ t = 2x \end{array} \right/ =$

$= \lim\limits_{x\to 0} \dfrac{2x - \frac{(2x)^3}{3!} + O(x^5) - 2x}{x\left(1 - \left(1 - \frac{x^2}{2!} + O(x^4)\right)\right)} =$

$= \lim\limits_{x\to 0} \dfrac{-\frac{8x^3}{6} + O(x^5)}{\frac{x^3}{2} + O(x^5)} =$

$= \lim\limits_{x\to 0} \dfrac{\cancel{x^3}\left(-\frac{4}{3} + \overbrace{O(x^2)}^{\to 0}\right)}{\cancel{x^3}\left(\frac{1}{2} + \underbrace{O(x^2)}_{\to 0}\right)} =$

$= \lim\limits_{x\to 0} \dfrac{-\frac{4}{3} + \overbrace{O(x^2)}^{\to 0}}{\frac{1}{2} + \underbrace{O(x^2)}_{\to 0}} = -\dfrac{8}{3}$

Svar: $-\dfrac{8}{3}$

4 b $\lim\limits_{x\to 1} \dfrac{(x^{\frac{1}{3}} - 1)(x-1)}{\arctan(2(x-1))^2} =$

$= \left/ \begin{array}{l} t = x - 1 \\ x \to 1 \Rightarrow \\ t \to 0 \end{array} \right/ = \lim\limits_{t\to 0} \dfrac{\left((1+t)^{\frac{1}{3}} - 1\right) t}{\arctan(2t)^2} =$

$= \lim\limits_{t\to 0} \dfrac{\left(1 + \frac{1}{3}t + O(t^2) - 1\right) t}{4t^2 + O(t^6)} =$

$= \lim\limits_{t\to 0} \dfrac{\frac{1}{3}t^2 + O(t^3)}{4t^2 + O(t^6)} =$

$= \lim\limits_{t\to 0} \dfrac{\cancel{t^2}\left(\frac{1}{3} + \overbrace{O(t)}^{\to 0}\right)}{\cancel{t^2}\left(4 + \underbrace{O(t^4)}_{\to 0}\right)} = \dfrac{1}{12}$

Svar: $\dfrac{1}{12}$

4 c. $$\int_0^{\infty} \frac{1}{4+9x^2}\,dx$$

Bilda. $$\int_0^{\omega} \frac{1}{4+9x^2}\,dx = \frac{1}{4}\int_0^{\omega} \frac{1}{1+\left(\frac{3x}{2}\right)^2}\,dx$$

$$= \frac{1}{4}\left[\frac{\arctan\left(\frac{3x}{2}\right)}{\frac{3}{2}}\right]_0^{\omega} =$$

$$= \frac{1}{6}\Big(\underbrace{\arctan\left(\frac{3\omega}{2}\right)}_{\to \pi/2} - \underbrace{\arctan 0}_{=0}\Big) \to \frac{\pi}{12}$$

då $\omega \to \infty$

Svar: Konv.

$$\int_0^{\infty} \frac{1}{4+9x^2}\,dx = \frac{\pi}{12}$$

5 $$y'' - 3y' + 2y = (2x+5)e^{3x}$$

1. Sök y_h $\quad y'' - 3y' + 2y = 0$

K.E. $r^2 - 3r + 2 = 0$

$r_1 = 1$, $r_2 = 2$

$$y_h = Ce^{x} + De^{2x}$$

2. Sök y_p $\quad$ Sub. $y = ze^{3x}$

$$y' = z'e^{3x} + 3ze^{3x} = (z' + 3z)e^{3x}$$

$$y'' = (z'' + 3z')e^{3x} + 3(z' + 3z)e^{3x} = (z'' + 6z' + 9z)e^{3x}$$

insättning ger

$$z'' + 6z' + 9z - 3(z' + 3z) + 2z = 2x + 5$$

$$z'' + 3z' + 2z = 2x + 5$$

$$z_p = ax + b, \quad z_p' = a, \quad z_p'' = 0$$

$$3a + 2(ax+b) = 2x + 5 \text{ ger } \begin{cases} a = 1 \\ b = 1 \end{cases}$$

$$y_p = (x+1)e^{3x}$$

Svar: $y = y_h + y_p = Ce^{x} + De^{2x} + (x+1)e^{3x}$

6. $y(x) + \int_0^x \frac{y(t)}{1+t^2}\,dt = 2\arctan x$

derivering ger

$$y'(x) + \frac{1}{1+x^2}\,y(x) = \frac{2}{1+x^2}$$

$$\Big/ \; i.f = e^{\int \frac{1}{1+x^2}dx} = e^{\arctan x} \; \Big/$$

$$y(x)e^{\arctan x} = \int \frac{2}{1+x^2} e^{\arctan x}\,dx$$

$$y(x)e^{\arctan x} = 2e^{\arctan x} + C$$

$$y(x) = 2 + Ce^{-\arctan x}$$

$x=0$ ger

$$y(0) + \underbrace{\int_0^0 \frac{y(t)}{1+t^2}\,dt}_{=0} = \underbrace{2\arctan 0}_{=0}$$

$y(0) = 0$ ger $C = -2$

Svar: $y(x) = 2 - 2e^{-\arctan x}$

7. $r = \frac{3}{7}\varphi^2, \quad 0 \le \varphi \le a.$

$$L = \int\sqrt{dx^2+dy^2} = \int\sqrt{\left(\frac{dx}{d\varphi}\right)^2+\left(\frac{dy}{d\varphi}\right)^2}\,d\varphi$$

$$= \left/ \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ \text{där } r = r(\varphi) \end{array} \right/ = \dots = \int\sqrt{r^2+(r')^2}\,d\varphi \qquad r' = \frac{6}{7}\varphi$$

$$= \int_0^a \sqrt{\frac{9\varphi^4}{49}+\frac{36\varphi^2}{49}}\,d\varphi =$$

$$= \frac{3}{7}\int_0^a \underbrace{|\varphi|}_{=\varphi}\sqrt{\varphi^2+4}\,d\varphi = \Big/ \; t = \varphi^2+4 \text{ osv} \; \Big/$$

$$= \frac{3}{7}\left[\frac{(\varphi^2+4)^{3/2}}{\frac{3}{\cancel{2}}\cdot\cancel{2}}\right]_0^a = \frac{1}{7}\left((a^2+4)^{3/2} - 8\right) = 8$$

$\Rightarrow (a^2+4)^{3/2} = 64$ | ger $a^2 = 12$

$a^2+4 = 64^{2/3}$ | Svar: $a = \sqrt{12} = 2\sqrt{3}$

$a^2+4 = (2^6)^{2/3}$