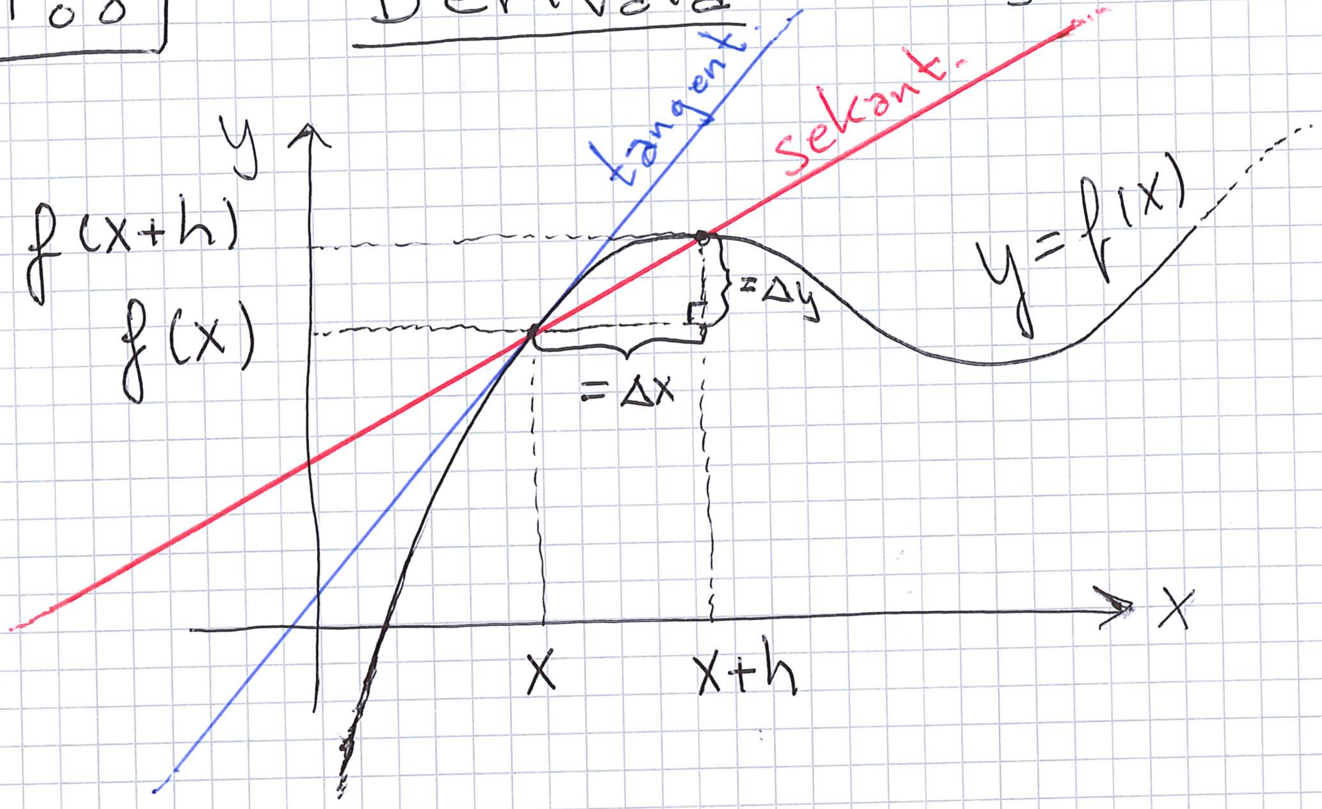
Fö 8 Derivata

Läs 4.1–4.3



$$\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}$$ = "sekantens lutning"

medelförändringen.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = y'(x)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

"f prim"

"tangentens lutning"
(k-värdet).

momentana förändringen
(ögonblickliga)

Def: f är deriverbar i en punkt a om

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ existerar ändligt.

(givet f def. i någon omgivning av a)

Gr.v kallas derivatan av f i a och skrivs $f'(a)$.

$$y = f(x)$$

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x)) =$$

$$= D(f(x))$$

Ex. Bestäm med derivatans definition $f'(x)$ då

$$f(x) = \sqrt{3x + 1}$$

L:
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \sqrt{3x+1}$$

obs!! parentes viktig.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} =$$

$$= \frac{\left(\sqrt{3(x+h)+1} - \sqrt{3x+1}\right)\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)}$$

$$= \frac{3(x+h)+1 - (3x+1)}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} =$$

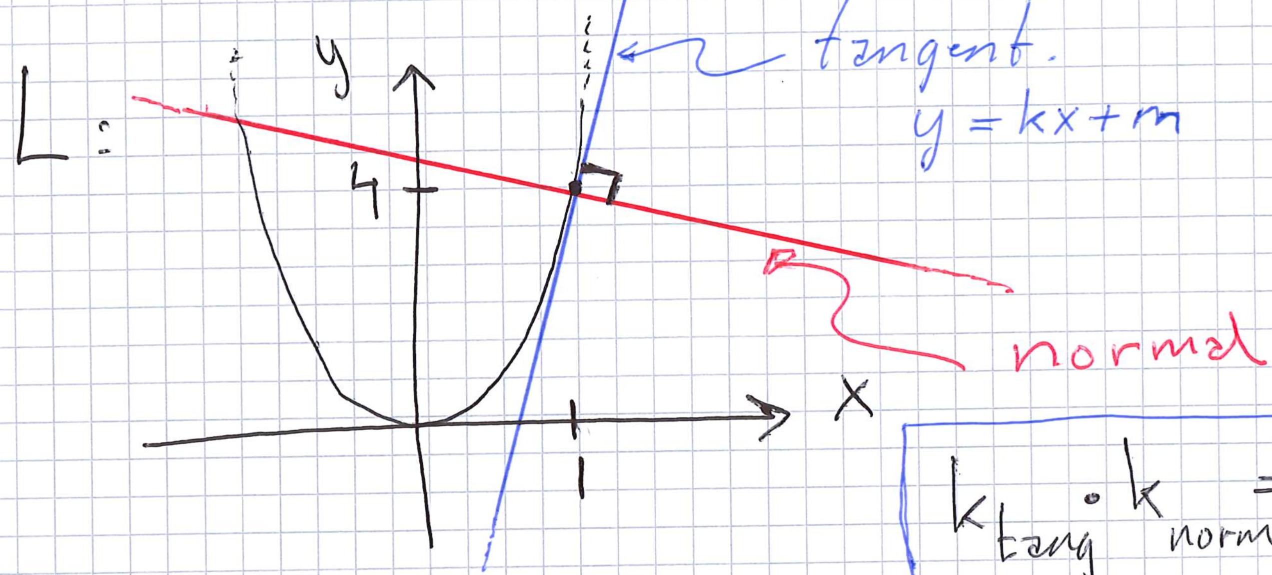
$$= \frac{\cancel{3x} + 3h + \cancel{1} - \cancel{3x} - \cancel{1}}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} =$$

$$= \frac{3}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \longrightarrow \frac{3}{2\sqrt{3x+1}}$$

då $h \to 0$

Svar: $f'(x) = \dfrac{3}{2\sqrt{3x+1}}$

Ex. Bestäm ekvationen för tangenten och normalen till $f(x) = 4x^2$ i punkten $(1,4)$



$$\boxed{k_{tang} \cdot k_{normal} = -1}$$

$$k_{tangent} = f'(1)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{4(x+h)^2 - (4x^2)}{h} = \lim_{h \to 0} \frac{8xh + 4h^2}{h} =$$

$$= \lim_{h \to 0} (8x + \underbrace{4h}_{\to 0}) = 8x$$

$$k_{tang} = f'(1) = 8 \cdot 1 = 8$$

forts. tangentens ekv.

$$y = 8x + m$$

(1,4) tillhör linjen.

$$4 = 8 \cdot 1 + m \iff m = -4$$

$$\boxed{y = 8x - 4}$$

normalens ekv. $k_{normal} = -\frac{1}{8}$

$$y = -\frac{1}{8}x + m \quad (\text{nytt } m).$$

$$4 = -\frac{1}{8} \cdot 1 + m_n \iff m_n = \frac{33}{8}$$

$$\boxed{y = -\frac{1}{8}x + \frac{33}{8}}$$

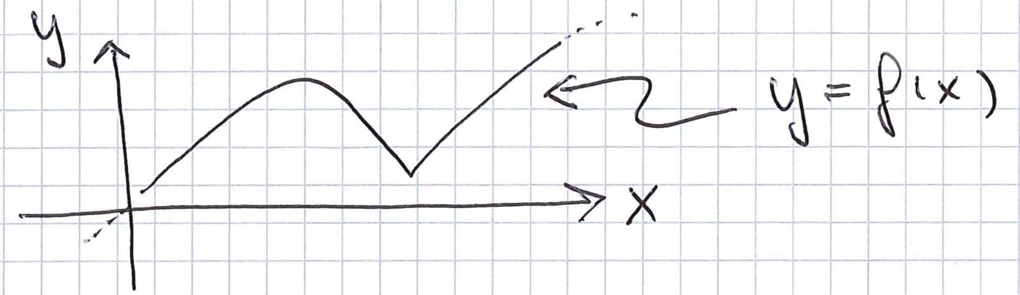
Svar: tang. ekv. $\boxed{y = 8x - 4}$

norm. ekv. $\boxed{y = -\frac{1}{8}x + \frac{33}{8}}$

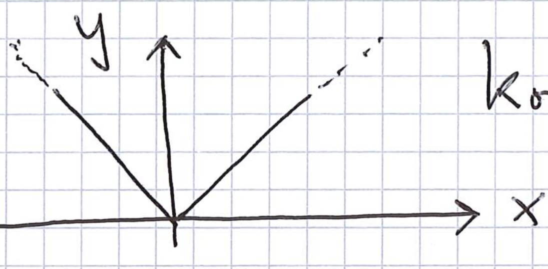
Sats: f deriverbar i a

$$\Longrightarrow \qquad \text{obs!!} \ \cancel{\Longleftarrow}$$

f kontinuerlig i a.



Ex. $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



kont. men ej deriverbar i $x = 0$.

Bevis av satsen ovan.

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \underbrace{\frac{f(x) - f(a)}{x - a}}_{\to f'(a)} (x - a) =$$

$$= f'(a) \cdot 0 = 0$$

dvs $\lim_{x \to a} f(x) = f(a)$.

Deriveringsregler. $f(x)$, $g(x)$.

$$(f+g)' = f' + g'$$

$$(cf)' = cf' \quad , \quad c = \text{konstant}.$$

$$(f \cdot g)' = f'g + fg' \qquad \boxed{\text{Produktregeln.}}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad \boxed{\text{Kvotregeln}}$$

Sammansatt funktion

Glöm inte mult. med inre derivatan

$$Df(g(x)) = f'(g(x)) \cdot \underbrace{g'(x)}_{\text{inre derivatan}}$$

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listan:

y	y'
x^{α}	$\alpha x^{\alpha-1}$, $\alpha \in \mathbb{R}$.
e^x	e^x
$\sin x$	$\cos x$

forts. →

y	y'
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$, $x>0$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$, $x>0$
$\tan x$	$\frac{1}{\cos^2 x} = \tan^2 x + 1$
$\arctan x$	$\frac{1}{1+x^2}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$

m.fl.

"Bevis"

$$De^x = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} e^x \Big(\underbrace{\frac{e^h-1}{h}}_{\to 1} \Big) = e^x$$

standardgr.v

$$y = \ln x \iff x = e^y$$

$$y' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

$$D\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$= \Big/ \text{ trig. formel } \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \Big/$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} =$$

$$\lim_{h\to 0} \underbrace{\cos\left(x+\frac{h}{2}\right)}_{\to \cos x} \cdot \underbrace{\frac{\sin\frac{h}{2}}{\frac{h}{2}}}_{\to 1 \text{ standardgr.v}} = \cos x$$

$$D x^\alpha = \frac{d}{dx}(x^\alpha) = \frac{d}{dx}\left(e^{\ln x^\alpha}\right) =$$

$$= \frac{d}{dx} e^{\alpha \ln x} = e^{\alpha \ln x} \cdot \underbrace{\frac{\alpha}{x}}_{\text{inre derivatan}} =$$

$$= e^{\ln x^\alpha} \cdot \frac{\alpha}{x} = x^\alpha \cdot \frac{\alpha}{x} = \alpha x^{\alpha - 1}$$

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fler bevis se boken.

Ex. Räkna ut derivatan av

$$\sqrt{3x+1}$$

L: $f(x) = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(3x+1)^{-\frac{1}{2}} \cdot 3 = \frac{3}{2\sqrt{3x+1}}$$

inre derivatan.

Alt. $u = 3x+1$

$f(u) = \sqrt{u}$ där $u = u(x)$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

kedjeregeln

inre derivatan.

$$\frac{df}{du} = \frac{1}{2\sqrt{u}} \qquad \frac{du}{dx} = 3$$

$$\frac{df}{dx} = \frac{1}{2\sqrt{u}} \cdot 3 = \frac{3}{2\sqrt{3x+1}}$$

Ex. $y = \sin(x^2+1)$

$y' = \cos(x^2+1) \cdot 2x = 2x\cos(x^2+1)$

Tänk "utifrån och in"

Ex. $y = e^{5x}$

$y' = e^{5x} \cdot 5 = 5e^{5x}$

Ex. $y = \sin^2 x$

$y' = 2\sin x \cos x = \sin 2x$

Ex. $y = e^{\sqrt{x^2+1}}$

$y' = e^{\sqrt{x^2+1}} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x =$ ...... förenkla.

Tänk.

utifrån och in.

$\frac{d}{dx}\sqrt{x^2+1} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x$

Ex. $$f(x) = e^{x}\cos 2x$$

$$f'(x) = e^{x}\cos 2x + e^{x}(-\sin 2x)\cdot 2 =$$
$$= e^{x}(\cos 2x - 2\sin 2x)$$

produktregeln.
+ kedjeregeln.

Ex. $$f(x) = \frac{\sqrt{x}}{(x+2)^2}$$

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+2)^2 - \sqrt{x}\,2(x+2)}{(x+2)^4} =$$

kvotregeln

$$= \frac{(x+2)\left(\frac{1}{2\sqrt{x}}(x+2) - 2\sqrt{x}\right)}{(x+2)^4} =$$

$$= \frac{x+2-4x}{2\sqrt{x}(x+2)^3} = \frac{2-3x}{2\sqrt{x}\,(x+2)^3}$$

Ex. $f(x) = 10^x = e^{\ln 10^x} = e^{x \ln 10}$

$f'(x) = e^{x \ln 10} \cdot \ln 10 = \ln 10 \cdot 10^x$

Ex. $f(x) = \ln \dfrac{(x+1)^2}{\sqrt{3x+1}}$

obs! Förenkla först. Derivera sedan.

$f(x) = 2 \ln(x+1) - \frac{1}{2} \ln(3x+1)$

$f'(x) = 2 \cdot \dfrac{1}{x+1} - \dfrac{1}{2} \cdot \dfrac{1}{3x+1} \cdot 3 =$

inre derivatan

$=$ ________

förenkla.

Derivatan av invers funktion

$$Df^{-1}(b) = \frac{1}{f'(a)} \quad , \quad f'(a) \neq 0 .$$

$$b = f(a)$$

Se sid 189.

$$y = f(x) \iff x = f^{-1}(y)$$

f injektiv.

$$Df^{-1}(y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f'(x)}$$

Ex. Visa att $f(x) = x^3 + x + 2$ har en deriverbar invers funktion f^{-1}. Beräkna $(f^{-1})'(12)$

L: f strängt väx. $f'(x) = 3x^2 + 1 > 0$

$$(f^{-1})'(12) = \frac{1}{f'(2)} = \frac{1}{3 \cdot 2^2 + 1} = \frac{1}{13}$$

$12 = x^3 + x + 2$

$\iff \quad x = 2$

Ex. Visa att $D \arctan x = \dfrac{1}{1+x^2}$

Bevis: $y = \arctan x \quad , \quad -\pi/2 < y < \pi/2$

$x = \tan y$

$$D \arctan x = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} =$$

$$\overset{*}{=} \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1} = \frac{1}{1+x^2}$$

v.s.b.

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$$x = \tan y = \frac{\sin y}{\cos y}$$

$$\frac{dx}{dy} = \frac{\cos y \cos y - \sin y(-\sin y)}{\cos^2 y} = 1 + \tan^2 y$$

Kvotregeln.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Ex. $f(x) = \arctan(e^{\sqrt{x}})$

$$f'(x) = \frac{1}{1+(e^{\sqrt{x}})^2} \cdot e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

Implicit derivering.

Ex. $y^3 + (x^2+1)y = x$ definierar y som funktion av x.

Räkna ut $y'(x)$.

L: $y^3 + (x^2+1)y = x$

derivera med avseende på x.

$$3y^2 \cdot y' + 2xy + (x^2+1)y' = 1$$

$$y' = \frac{1-2xy}{3y^2+x^2+1}$$
