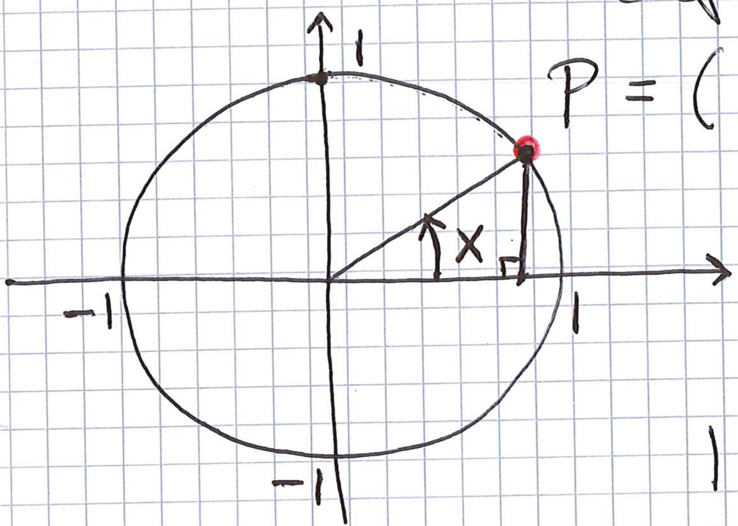
Fö 3 Läs 2.4.

Trigonometri via enhetscirkeln

Definition:



$P = (\cos x, \sin x)$

x = vridningen och mäts i radianer

1 varv = $360^\circ = 2\pi$ rad.

moturs är **positiv led**.

π = "halvt varv".

$\frac{\pi}{2} = 90^\circ$ rät vinkel.

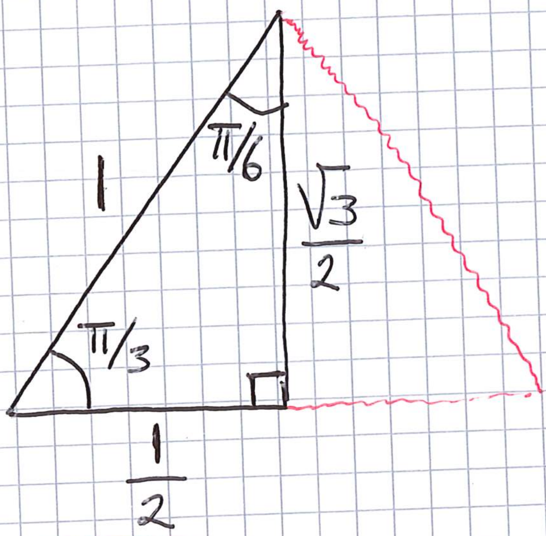
Ex. $\cos \frac{\pi}{2} = 0$

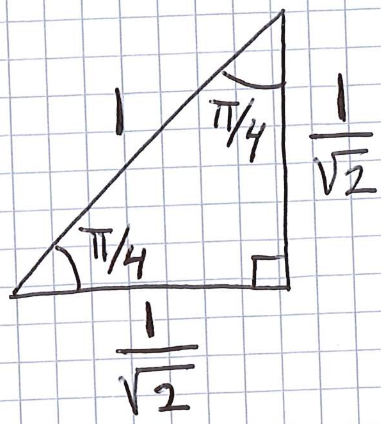
$\sin \frac{3\pi}{2} = \sin\left(-\frac{\pi}{2}\right) = -1$

$\sin \pi = 0$

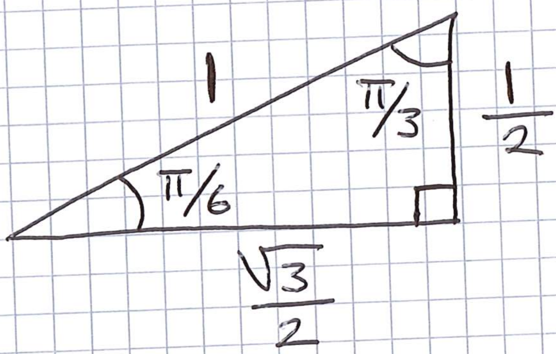
$\cos \pi = -1$

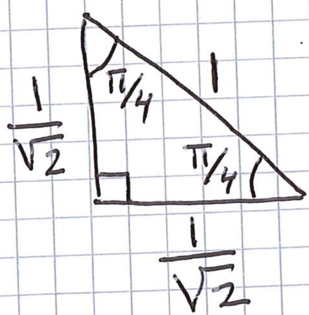
$\frac{\pi}{4} = 45^\circ$ $\quad \frac{\pi}{3} = 60^\circ$ $\quad \frac{\pi}{6} = 30^\circ$





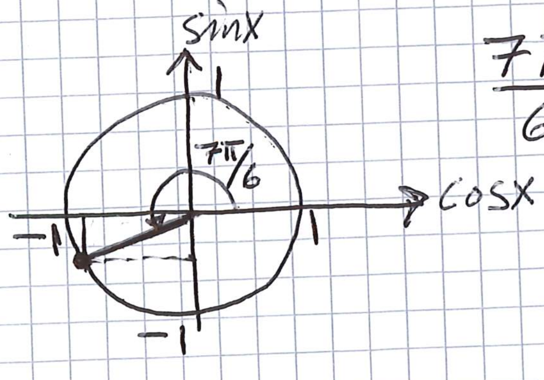
Jätteviktiga.





Ex. Beräkna $\sin\frac{7\pi}{6}$ Obs!

L: $\boxed{\text{Rita alltid enhetscirkeln}}$.

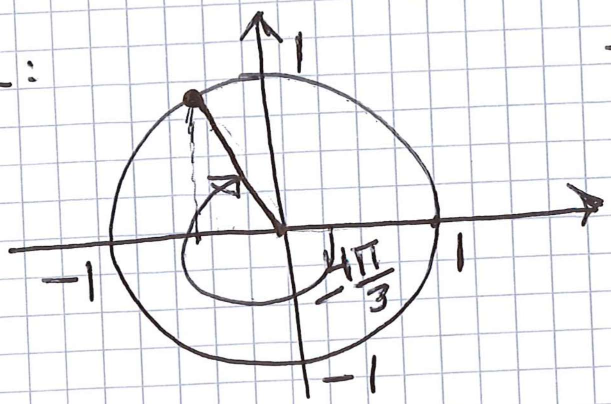


$$\frac{7\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \pi + \frac{\pi}{6}$$

$$\sin\frac{7\pi}{6} = -\frac{1}{2}$$

Ex. Beräkna $\cos\left(-\frac{4\pi}{3}\right)$

L:



$$-\frac{4\pi}{3} = -\pi - \frac{\pi}{3}$$

$$\cos\left(-\frac{4\pi}{3}\right) = -\frac{1}{2}$$

Def: $\tan x = \frac{\sin x}{\cos x}$, $x \neq \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

$\cot x = \frac{\cos x}{\sin x}$, $x \neq n\pi$, $n \in \mathbb{Z}$.

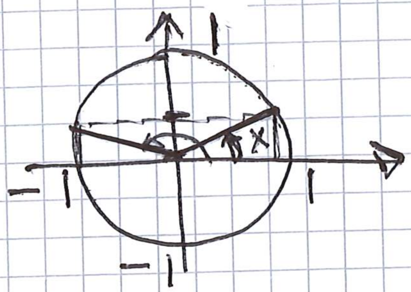
Ex. $\tan\frac{5\pi}{4} = \frac{\sin\frac{5\pi}{4}}{\cos\frac{5\pi}{4}} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$

Ex. $\cot\frac{2\pi}{3} = \frac{\cos\frac{2\pi}{3}}{\sin\frac{2\pi}{3}} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

Ekvationer

Ex. Lös ekvationen $\sin x = \frac{1}{2}$

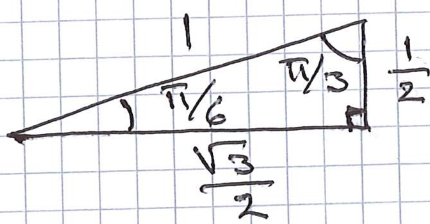
L. Rita enhetscirkeln. Kom ihåg!



$$\sin x = \frac{1}{2}$$

$$\iff$$

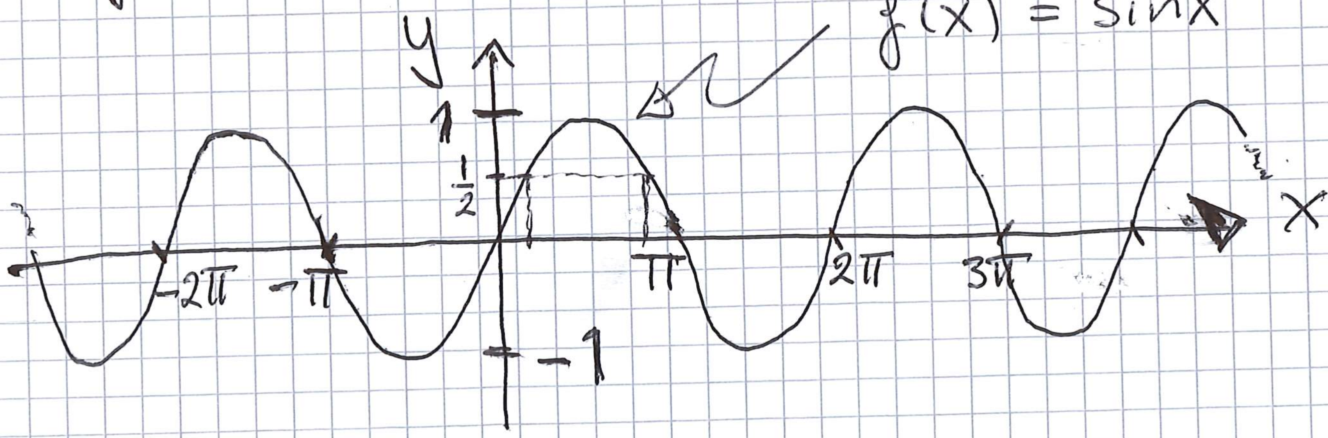
$$x = \begin{cases} \pi/6 + n2\pi \\ \pi - \frac{\pi}{6} + n2\pi \end{cases}, \quad n \in \mathbb{Z}.$$



Svar: $x = \begin{cases} \pi/6 + n2\pi \\ 5\pi/6 + n2\pi \end{cases}$

n heltal.

Graf av $f(x) = \sin x$

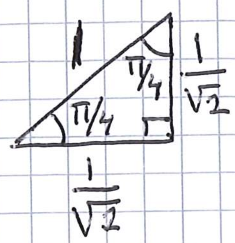


$V_f = [-1, 1]$, $D_f: x \in \mathbb{R}$.

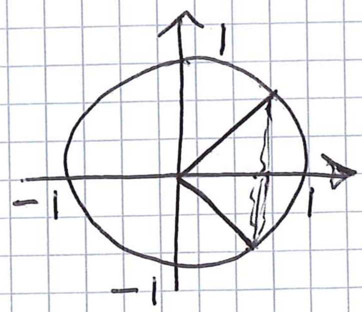
Ex. Lös ekvationen $\cos\left(2x+\frac{\pi}{6}\right)=\frac{1}{\sqrt{2}}$

L. $\cos\left(2x+\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$

/ Sub. $t = 2x + \frac{\pi}{6}$ /



$$\cos t = \frac{1}{\sqrt{2}}$$



$$\Longleftrightarrow$$

$$t = \pm\frac{\pi}{4} + n2\pi \qquad n \in \mathbb{Z}.$$

Obs! Perioden skall på direkt.

$$\Longleftrightarrow$$

$$2x+\frac{\pi}{6} = \begin{cases} \frac{\pi}{4} + n2\pi \\ -\frac{\pi}{4} + n2\pi \end{cases}$$

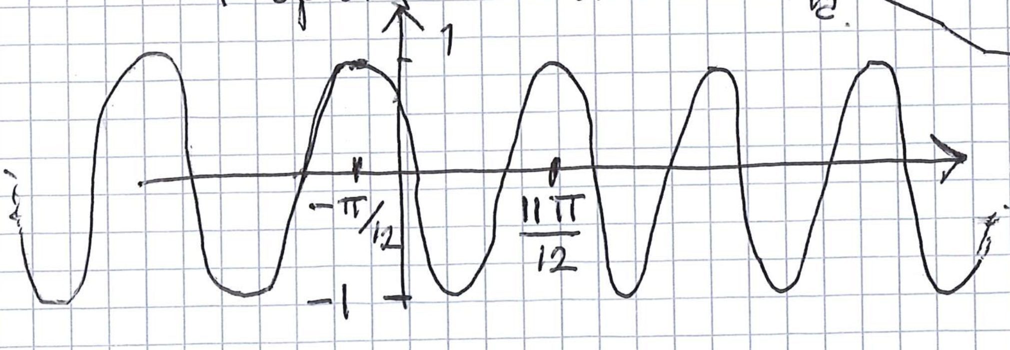
$$\Longleftrightarrow$$

Svar:

$$x = \begin{cases} \frac{\pi}{24} + n\pi \\ -\frac{5\pi}{24} + n\pi \end{cases}, \quad n \text{ heltal}.$$

Obs! Perioden ändras vid division med 2.

ihoptryckt cosinuskurva.



$f(x) = \cos\left(2x+\frac{\pi}{6}\right)$

Period $= \pi$.

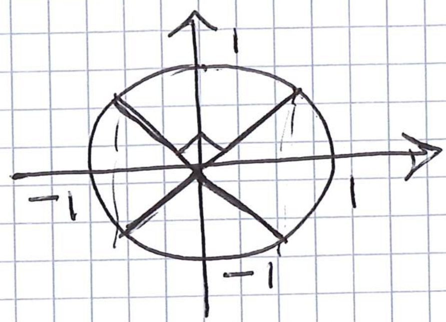
Ex. Lös ekvationen $\cos^2 2x = \frac{1}{2}$

L: $\cos^2 2x = \frac{1}{2}$ / Sub. $t = 2x$ /

$\cos^2 t = \frac{1}{2} \Leftrightarrow \cos t = \pm \frac{1}{\sqrt{2}}$

$\Leftrightarrow t = \frac{\pi}{4} + n\frac{\pi}{2}$

$n \in \mathbb{Z}$.



$\Leftrightarrow 2x = \frac{\pi}{4} + n\frac{\pi}{2}$

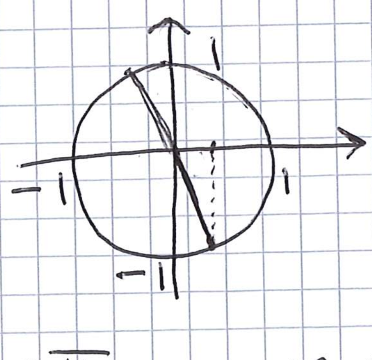
$\Leftrightarrow x = \frac{\pi}{8} + \frac{n\pi}{4}, \quad n \in \mathbb{Z}$.

Ex. Lös ekv. $\tan x = -\sqrt{3}$

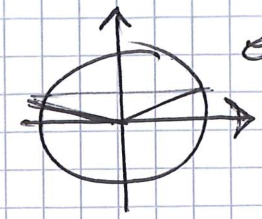
L: $\frac{\sin x}{\cos x} = -\sqrt{3} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$

$\Leftrightarrow x = -\frac{\pi}{3} + n\pi$

, n heltal.



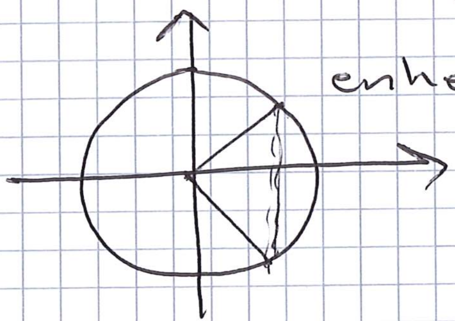
Svar: $x = \frac{2\pi}{3} + n\pi, \quad n \in \mathbb{Z}$.



Allmänt:

$$\sin x = \sin y \iff x = \begin{cases} y + n2\pi \\ \pi - y + n2\pi \end{cases}, \ n \text{ heltal.}$$

$$\cos x = \cos y \iff \left\{ x = \pm y + n2\pi \right., \ n \text{ heltal.}$$



$$\tan x = \tan y \iff x = y + n\pi, \ n \text{ heltal.}$$

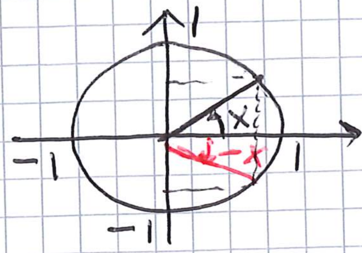
Formler

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

trig. ettan

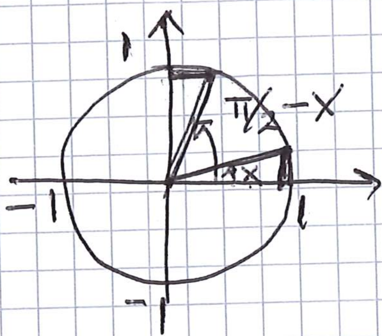
$$\boxed{\sin(-x) = -\sin x}$$

$$\boxed{\cos(-x) = \cos x}$$



$$\boxed{\sin\left(\frac{\pi}{2} - x\right) = \cos x}$$

$$\boxed{\sin x = \cos\left(\frac{\pi}{2} - x\right)}$$



Additionslagar

$$\begin{cases} \cos(x-y) = \cos x \cos y + \sin x \sin y \\ \cos(x+y) = \cos x \cos y - \sin x \sin y \\ \sin(x+y) = \sin x \cos y + \cos x \sin y \\ \sin(x-y) = \sin x \cos y - \cos x \sin y \end{cases}$$

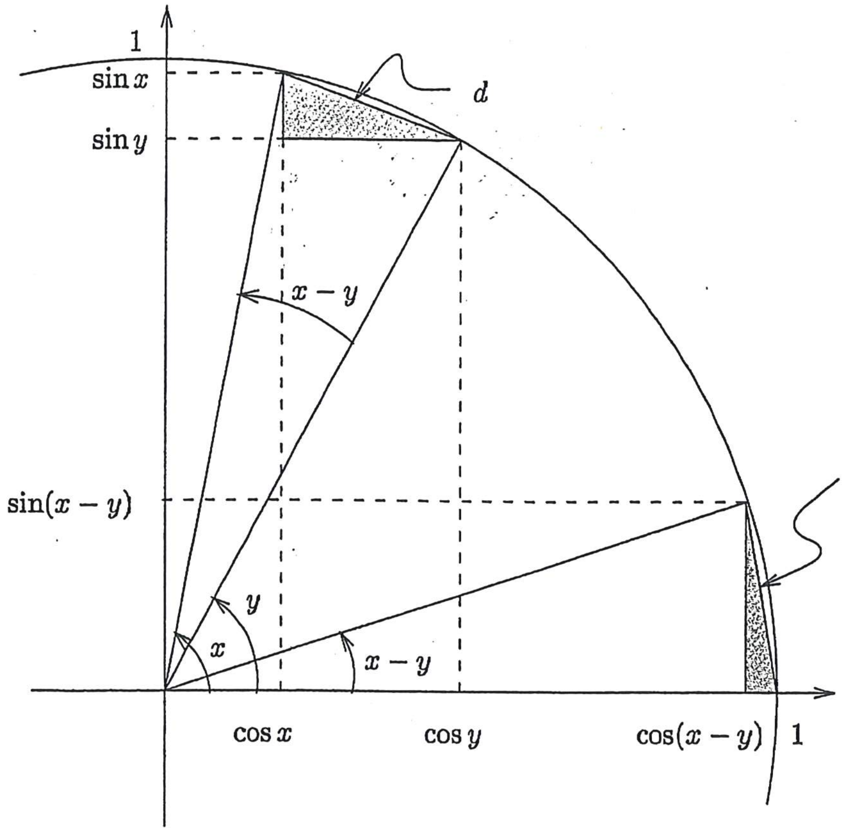
$$\boxed{\sin 2x = 2 \sin x \cos x}$$

$$\boxed{\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1}$$

$$\boxed{\sin^2 x = \frac{1 - \cos 2x}{2}}$$

$$\boxed{\cos^2 x = \frac{1 + \cos 2x}{2}}$$

Bevis Titta på nedanstående figur.



Vi ser två likadana likbenta trianglar utritade, toppvinkeln hos dem är $x - y$. Då måste "baserna" i dessa trianglar vara lika långa, säg att längden är d . Dessa baser utgör dock hypotenusan i de två rätvinkliga skuggade trianglarna, så med hjälp av Pythagoras sats erhålls sambandet

$$(\cos y - \cos x)^2 + (\sin x - \sin y)^2 = d^2 = (1 - \cos(x - y))^2 + (\sin(x - y))^2$$

och om vi utvecklar detta och använder trigonometriska ettan så får vi direkt att

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

vilket skulle visas.

□

Ex Visa att $\cos 3x = 4\cos^3 x - 3\cos x$

Bevis: $V.L = \cos 3x = \cos(x + 2x) =$

$= \cos x \cos 2x - \sin x \sin 2x =$

$= \cos x (2\cos^2 x - 1) - \sin x (2\sin x \cos x) =$

$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x =$

$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x =$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x =$

$= 4\cos^3 x - 3\cos x = H.L$

v.s.b.

Ex. Lös ekv. $2\cos^2 x - \sin x = 1$

L: $2\cos^2 x - \sin x = 1$ trig. ettan

$2(1-\sin^2 x) - \sin x = 1$

$-2\sin^2 x - \sin x + 1 = 0$

/ Sub. $t = \sin x$, $-1 \le t \le 1$ /

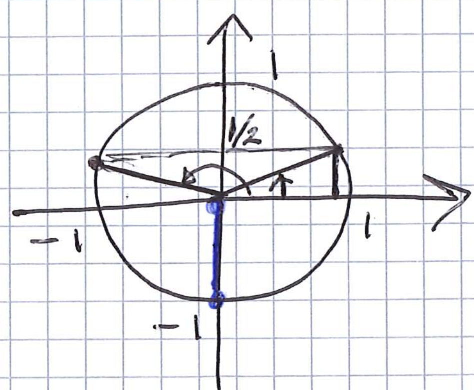
$-2t^2 - t + 1 = 0$

$t^2 + \frac{1}{2}t - \frac{1}{2} = 0$

$t = -\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}} = -\frac{1}{4} \pm \frac{3}{4} = \begin{cases} 1/2 \\ -1 \end{cases}$

$\sin x = \begin{cases} 1/2 \\ -1 \end{cases}$

$\Longleftrightarrow$



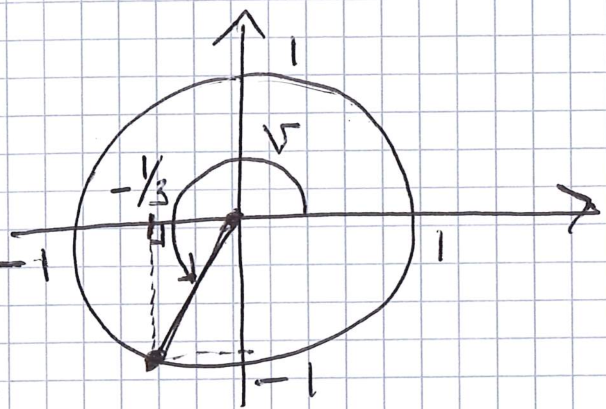
Svar: $x = \frac{\pi}{6} + n\frac{2\pi}{3}$, $n \in \mathbb{Z}$.

Alt: $x = \begin{cases} \pi/6 + n2\pi \\ 5\pi/6 + n2\pi \\ \frac{3\pi}{2} + n2\pi \end{cases}$, $n \in \mathbb{Z}$

Ex. Bestäm $\sin v$, $\tan v$, $\cot v$ då $\cos v = -\frac{1}{3}$ och

$\pi < v < \frac{3\pi}{2}$.

L: Rita enhetscirkeln.



$$\sin^2 v + \left(\frac{1}{3}\right)^2 = 1^2$$ trig. ettan

$$\sin v = (\pm)\sqrt{\frac{8}{9}} = (\pm)\frac{2\sqrt{2}}{3}$$

$$\sin v = -\frac{2\sqrt{2}}{3}$$

$$\tan v = \frac{-\frac{2\sqrt{2}}{3}}{-\frac{1}{3}} = 2\sqrt{2}$$

$$\cot v = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

Ex. Lös ekv. $\sin(2x+\frac{\pi}{3}) = \cos(x-\frac{\pi}{4})$

L: trig. formel $\boxed{\cos x = \sin(\frac{\pi}{2}-x)}$

ger

$$\sin(2x+\frac{\pi}{3}) = \cos(x-\frac{\pi}{4})$$

$$\Longleftrightarrow$$

$$\sin(2x+\frac{\pi}{3}) = \sin\left(\frac{\pi}{2}-\left(x-\frac{\pi}{4}\right)\right)$$

$$\sin(2x+\frac{\pi}{3}) = \sin\left(\frac{3\pi}{4}-x\right)$$

$$\Longleftrightarrow$$

$$2x+\frac{\pi}{3} = \begin{cases} \frac{3\pi}{4}-x + n2\pi \\ \pi-\left(\frac{3\pi}{4}-x\right)+n2\pi \end{cases}, n\in\mathbb{Z}.$$

$$\Longleftrightarrow$$

Svar: $$x = \begin{cases} \frac{5\pi}{36} + \frac{n2\pi}{3} \\ -\frac{\pi}{12} + n2\pi \end{cases}, \quad n\in\mathbb{Z}$$