Fö 1 Läs 1,7 samt 2.6

## Komplexa tal

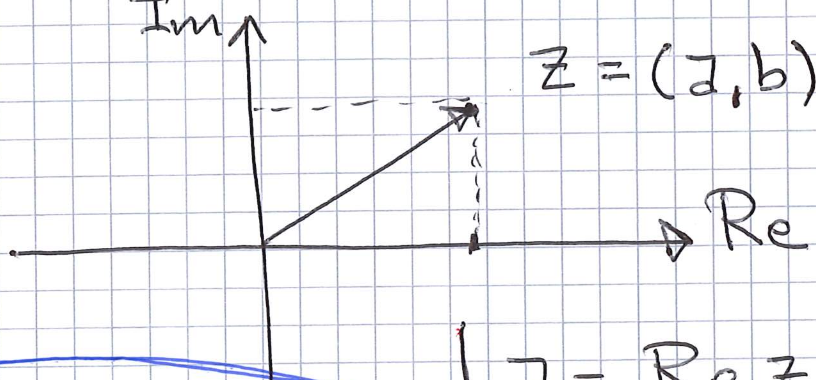
$x \in \mathbb{R}$.

$x^2 = -9$

jfr. $z^2 = -9$

$z \in \mathbb{C}$

### Det komplexa talplanet.




$z = (a, b) = a + ib$

$$\boxed{i^2 = -1}$$

Tänk $i$ som ett kommatecken

$a = \operatorname{Re} z$ "realdelen"

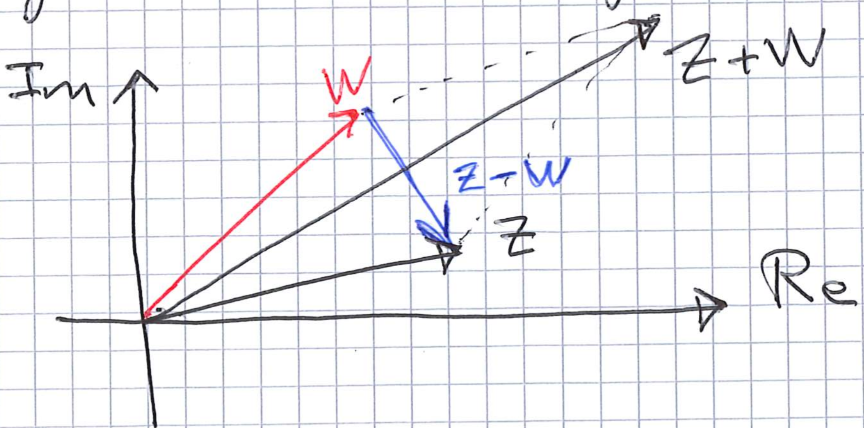
$b = \operatorname{Im} z$ "imaginärdelen"

Ex. $(3+i)(2-5i) + 7 - 2i =$

$$6 - 15i + 2i - 5\underbrace{i^2}_{=-1} + 7 - 2i = 18 - 15i$$

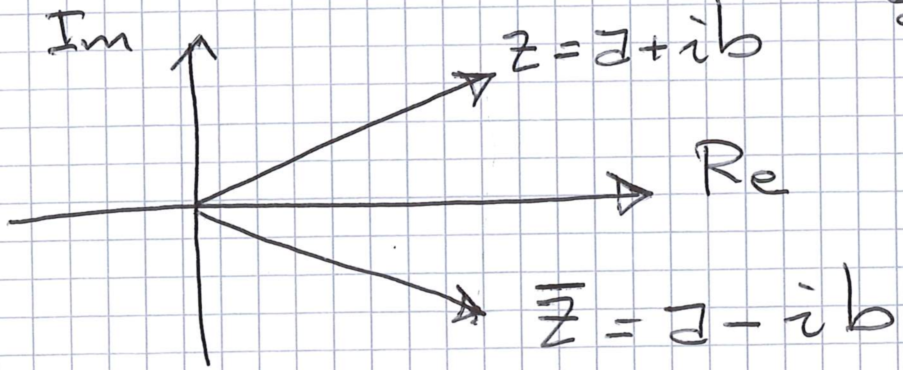
---

### Geometrisk tolkning.

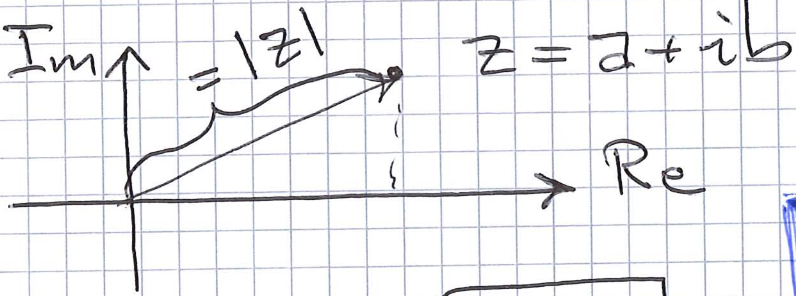



---

**Def:** Konjugering "spegling i reella axeln".



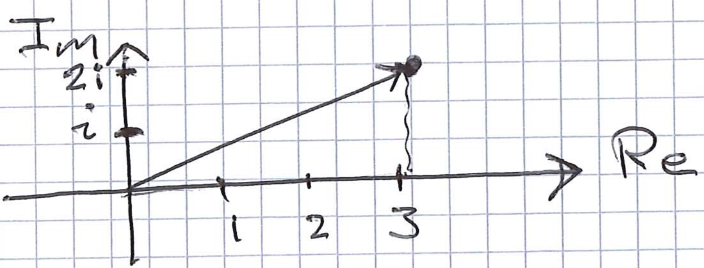

Absolutbelopp.




$$|z| = \sqrt{a^2 + b^2}$$

**Obs!** Blanda inte in $i$.

Ex. $|3 + 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}$




---

Allmänt: $\boxed{z\bar{z} = |z|^2}$

$z = x + iy$

$\bar{z} = x - iy$

Division: $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w|^2}$

Fler räkneregler: $\qquad z\bar{z} = |z|^2$

$$\overline{zw} = \bar{z}\,\bar{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

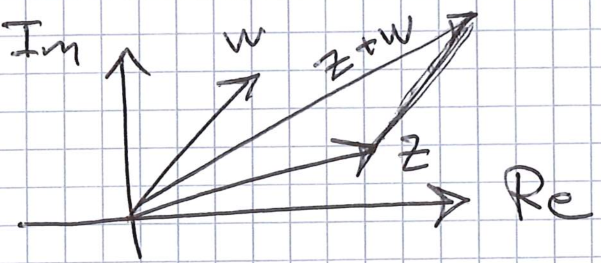
$$\overline{z+w} = \bar{z} + \bar{w}$$

$$|zw| = |z||w|$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

Bevis: Se boken.

men $\quad |z+w| \leq |z| + |w| \quad$ triangelolikheten.




Ex. $\qquad z = \dfrac{(1+i)^3}{(2+i)(5-i)}$

Beräkna $|z|$ dvs

bestäm absolutbeloppen av $z$.

L: $\quad |z| = \left|\dfrac{(1+i)^3}{(2+i)(5-i)}\right| = \dfrac{|1+i|^3}{|2+i||5-i|} =$

$$= \ldots\ldots = \frac{(\sqrt{2})^3}{\sqrt{5}\sqrt{26}} = \frac{2\cancel{\sqrt{2}}}{\sqrt{5}\cancel{\sqrt{2}}\sqrt{13}} = \frac{2\sqrt{5}\sqrt{13}}{65}$$

Ex. $\dfrac{1+i}{3+i} = \dfrac{(1+i)(3-i)}{(3+i)(3-i)} = \dfrac{3-i+3i-i^2}{9-i^2} =$

$= \dfrac{4+2i}{10} = \dfrac{2+i}{5} = \dfrac{2}{5} + \dfrac{i}{5}$

---

Ex. Lös ekv. $z + 3\bar{z} = 1+i$ *

L: Sätt $z = x+iy$ ger
$\bar{z} = x - iy$ i * ovan.

$x + iy + 3(x - iy) = 1+i$

$4x - 2iy = 1+i$

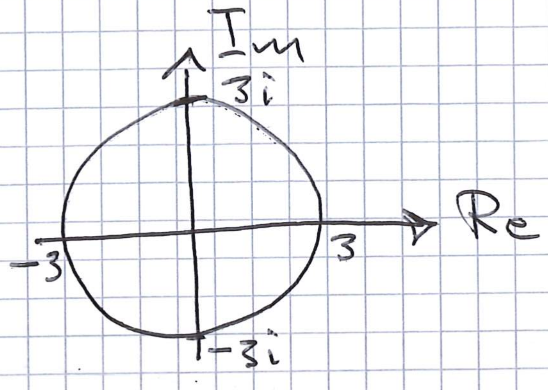
$\Leftrightarrow \begin{cases} 4x = 1 & \text{Re-del.} \\ -2y = 1 & \text{Im-del} \end{cases}$

$\Leftrightarrow \begin{array}{l} x = 1/4 \\ y = -1/2 \end{array}$

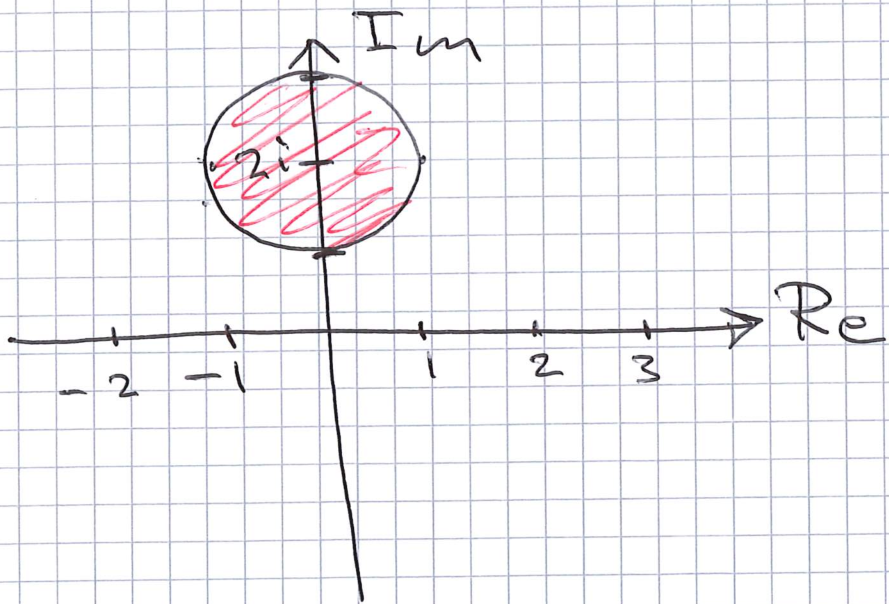
Svar: $z = \dfrac{1}{4} - \dfrac{i}{2}$

---

Ex. Rita $|z| = 3$




Ex Rita $|z - 2i| \leq 1$




Svårare: Sätt $z = x + iy$
räkna på.

# Andragradsekvationer.

$$x^2 + px + q = 0$$

$$x = -\frac{p}{2} \pm \sqrt{\left(-\frac{p}{2}\right)^2 - q}$$

Ex. Lös $z^2 + 4z + 13 = 0$

L: $z^2 + 4z + 13 = 0$

$$(z+2)^2 - 4 + 13 = 0$$

$$(z+2)^2 = -9$$

$$(z+2)^2 = 9i^2$$

$$z+2 = \pm 3i$$

$$z = -2 \pm 3i$$

Svar: $\begin{cases} z_1 = -2+3i \\ z_2 = -2-3i \end{cases}$

Ex. Lös $z^2 = 3+4i$

L: Sub. $z = x+iy$

$$(x+iy)^2 = 3+4i$$

$$x^2-y^2+2ixy = 3+4i$$

$$\Longleftrightarrow \begin{cases} x^2-y^2 = 3 & * \\ 2xy = 4 & \text{ger } y = \frac{2}{x} \end{cases}$$

sätts in i *.

$$x^2-\left(\frac{2}{x}\right)^2 = 3$$

$$x^4-4 = 3x^2$$

$$x^4-3x^2-4 = 0 \qquad / \text{Sub. } t=x^2, \ t \geq 0 /$$

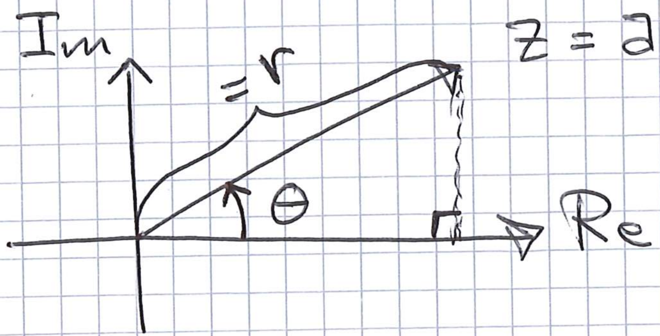
$$t^2-3t-4 = 0$$

$$\cancel{t_1 = -1}, \quad t_2 = 4 \quad \text{ger} \quad x = \pm 2$$

$$y = \pm 1$$

Svar: $\begin{cases} z_1 = 2+i \\ z_2 = -2-i \end{cases}$

# Polär form




$z = a + ib$

$\cos\theta = \dfrac{a}{r}$

$\sin\theta = \dfrac{b}{r}$

$r = |z|$

$\theta = \arg z$ ← ej entydigt bestämt. "hela varv".

normalform

$z = a+ib = r\cos\theta + i r\sin\theta =$

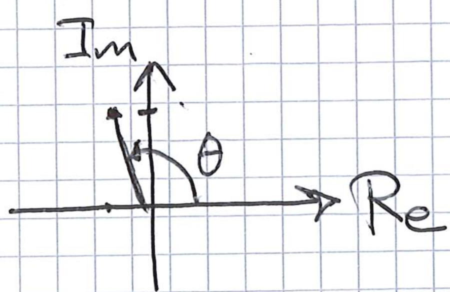
$= \underline{r(\cos\theta + i\sin\theta)} = re^{i\theta}$

Polär form.

Def: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Ex. $-1+\sqrt{3}i = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$

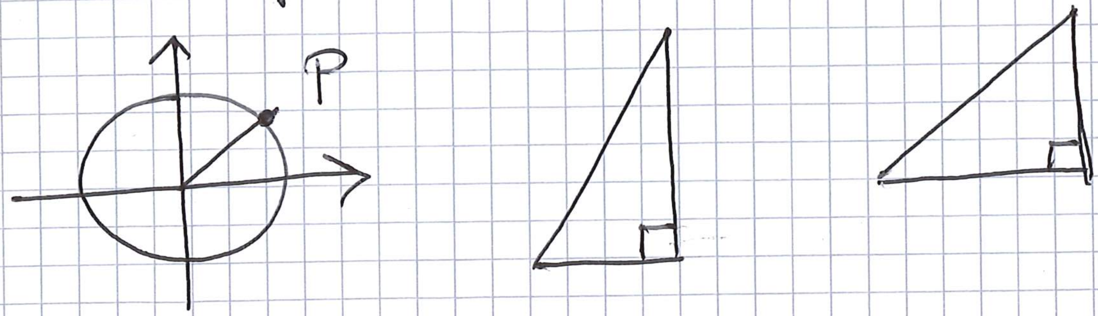
$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2e^{i\frac{2\pi}{3}}$




Rita alltid

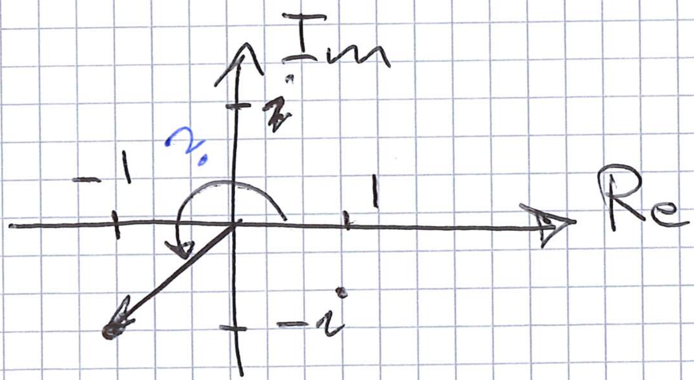
**Rep:** Se **Fö3** nollep.

trig. i enhetscirkeln.




---

Ex. $$-1-i = \sqrt{2}\,e^{i\frac{5\pi}{4}} = \sqrt{2}\,e^{i\left(-\frac{3\pi}{4}\right)} = \sqrt{2}\,e^{-i\frac{3\pi}{4}}$$




---

Ex. $$4e^{i\frac{7\pi}{6}} = 4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = -2\sqrt{3} - 2i$$

Rita!!

Viktiga regler.

Bevis: Se boken.

$$\boxed{e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}}$$

$$\boxed{\frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}}$$

de Moivrés formel.

$$\boxed{\left(e^{i\theta}\right)^n = e^{in\theta}}, \quad n \in \mathbb{N}.$$

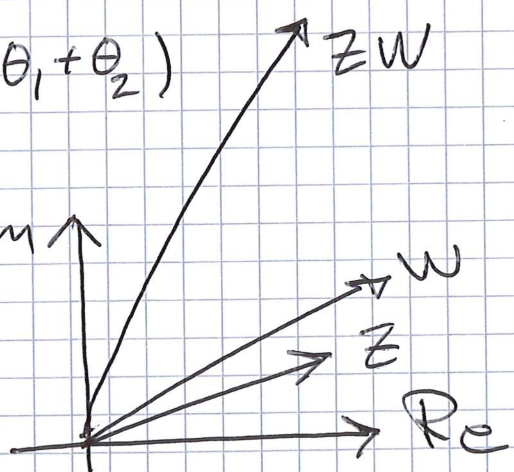
obs!

Ex. $z = r_1 e^{i\theta_1}$ $\quad w = r_2 e^{i\theta_2}$

$$zw = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$\arg(zw) = \arg z + \arg w$

$\arg\left(\frac{z}{w}\right) = \arg z - \arg w$




---

Ex. Bestäm $\arg\left(\frac{(1+i)^3}{(\sqrt{3}+i)(-2+2i)}\right)$

L: $3 \cdot \frac{\pi}{4} - \left(\frac{\pi}{6} + \frac{3\pi}{4}\right) = -\frac{\pi}{6}$, $k \in \mathbb{Z}$.

Alla argument är $-\pi/6 + k2\pi$

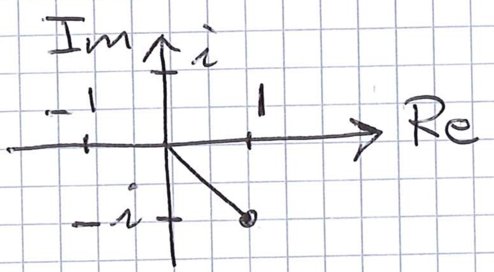
Ex. Beräkna $(1-i)^6$

L: $(1-i)^6 = \left(\sqrt{2}e^{i\frac{7\pi}{4}}\right)^6 =$

$= \left(2^{\frac{1}{2}}\right)^6 e^{i\frac{21\pi}{2}} = 2^3 e^{i\left(5\cdot\frac{4\pi}{2}+\frac{\pi}{2}\right)} = 8e^{i\frac{\pi}{2}} =$

$= \underline{\underline{8i}}$

$1-i = \sqrt{2}\,e^{i\left(-\frac{\pi}{4}\right)}$



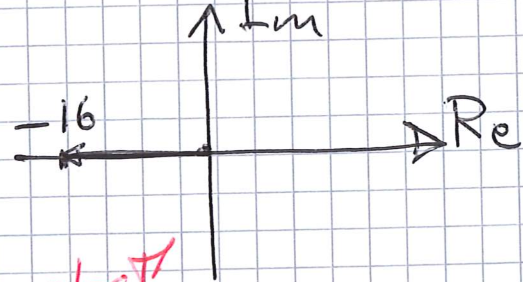

kontroll: $(1-i)(1-i) = -2i$

$(-2i)^3 = -8(-1)i = 8i$

Binomisk ekvation, $\boxed{z^n = w}$

$n \in \mathbb{Z}$.

Ex. Lös $z^4 = -16$




L: Sätt $z = re^{i\theta}$

$(re^{i\theta})^4 = 16e^{i\pi}$

$r^4 e^{i4\theta} = 16e^{i\pi}$

obs! Glöm inte $+k2\pi$.

$$\Longleftrightarrow \begin{cases} r^4 = 16 \\ 4\theta = \pi + k2\pi \end{cases} \Longleftrightarrow \begin{cases} r = 2 \\ \theta = \frac{\pi}{4} + \frac{k\pi}{2} \end{cases}$$

där $k = 0, 1, 2, 3$

$$z = 2e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)}, \quad k=0,1,2,3$$

$k=0$

$$z_1 = 2e^{i\frac{\pi}{4}} = 2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) =$$

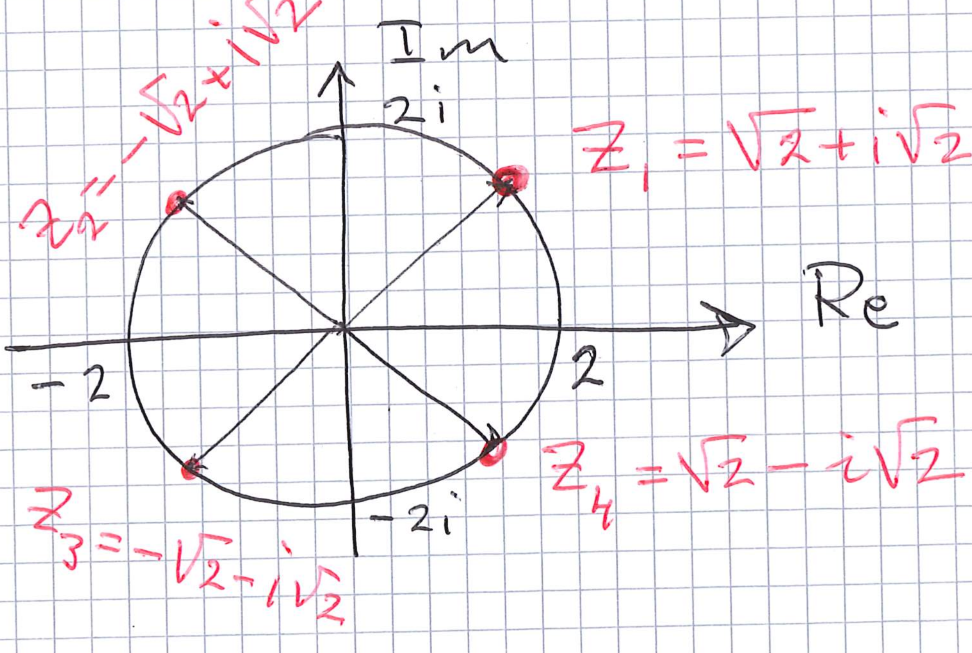
$$= 2\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = \sqrt{2}+i\sqrt{2}$$

$k=1$

$$z_2 = 2e^{i\frac{3\pi}{4}} = \ldots = -\sqrt{2}+i\sqrt{2}$$

$$z_3 = \ldots = -\sqrt{2}-i\sqrt{2}$$

$$z_4 = \sqrt{2}-i\sqrt{2}$$




Svar:
$$\begin{cases} z_1 = 2e^{i\pi/4} = \sqrt{2}+i\sqrt{2} \\ z_2 = \ldots = -\sqrt{2}+i\sqrt{2} \\ z_3 = \ldots = -\sqrt{2}-i\sqrt{2} \\ z_4 = \ldots = \sqrt{2}-i\sqrt{2} \end{cases}$$

# Eulers formler

$$\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}}$$

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

Se här: $e^{ix} = \cos x + i\sin x$

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$$

adderá ... osv

subtrahera ... osv

Pascals triangel.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Också bra.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

osv.

Binomialutveckling.

Ex. Visa att $\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$

Bevis: $V.L = \cos^3 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 =$

$$= \frac{(e^{ix}+e^{-ix})^3}{8} = \Big/ (a+b)^3 = a^3+3a^2b+3ab^2+b^3 \Big/ =$$

$$= \frac{1}{8}\left((e^{ix})^3 + 3(e^{ix})^2 e^{-ix} + 3e^{ix}(e^{-ix})^2 + (e^{-ix})^3\right) =$$

$$= \frac{1}{8}\left(e^{i3x} + 3e^{ix} + 3e^{-ix} + e^{-i3x}\right) =$$

$$= \frac{1}{4}\cdot\frac{e^{i3x}+e^{-i3x}}{2} + \frac{3}{4}\cdot\frac{e^{ix}+e^{-ix}}{2} =$$

$$= \frac{1}{4}\cos 3x + \frac{3}{4}\cos x \qquad \text{V.S.B.}$$