Fö 1 | Läs 5.1–5.2

Primitiva funktioner

Om $\frac{d}{dx}F(x) = f(x)$ så utgör $F(x)$ en primitiv funktion till $f(x)$.

Sats: Alla primitiver till $f(x)$ ges av $F(x)+C$, $C=$konstant

Bevis: Antag F, G primitiver till $f(x)$.

$$\frac{d}{dx}(G-F) = \frac{d}{dx}G - \frac{d}{dx}F = f-f=0$$

$$\Leftrightarrow \quad G-F = C \quad \Leftrightarrow \quad G = F+C.$$

Skrivsätt. $$\int f(x)\,dx = F(x)+C$$

Ex. $$\int \left(x^5 - \frac{3}{x^2} + \frac{1}{\sqrt{x}}\right) dx = \int \left(x^5 - 3x^{-2} + x^{-1/2}\right) dx =$$

$$= \frac{x^6}{6} - \frac{3x^{-1}}{-1} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C =$$

$$= \frac{1}{6}x^6 + \frac{3}{x} + 2\sqrt{x} + C$$

Obs! Kontrollera.

Ex. Bestäm __alla__ primitiva funktioner till $\dfrac{x^2+1}{\sqrt{x}}$

L: $\displaystyle\int \frac{x^2+1}{\sqrt{x}}\,dx = \int (x^{3/2} + x^{-1/2})\,dx =$

$\boxed{\text{Förenkla först}}$

$\displaystyle = \frac{x^{5/2}}{\frac{5}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}x^2\sqrt{x} + 2\sqrt{x} + C$

Ex. Bestäm __en__ primitiv funktion till

$\dfrac{5}{x^2+9}$

L: $\displaystyle\int \frac{5}{x^2+9}\,dx = \frac{5}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx =$

$\displaystyle = \frac{5}{9}\,\frac{\arctan\left(\frac{x}{3}\right)}{\frac{1}{3}} + C$

Svar: $\dfrac{5}{3}\arctan\left(\dfrac{x}{3}\right)$

Sammansatt funktion

Kompensera för inre derivatan är bara lätt när inre derivatan är en konstant.

Ex. $\int \sin 2x \, dx = \frac{-\cos 2x}{2} + C$

Ex. $\int e^{3x} \, dx = \frac{e^{3x}}{3} + C$

Ex. $\int \sqrt{1-x^2} \, dx = ?$ svårare uppgift.

Allmänt: $\int f(g(x)) \, dx = \frac{F(g(x))}{g'(x)} + C$

Obs! Sant endast om $g'(x) = \text{konstant}$

Bevis: $\frac{d}{dx} \frac{F(g(x))}{g'(x)} = \frac{f(g(x))g'(x)g'(x) - F(g(x))g''(x)}{(g'(x))^2} =$

$= f(g(x)) \iff g''(x) = 0$ dvs

$g'(x) = \text{konstant}.$

Variabelsub.

Ex. $$\int x(1+5x^2)^3\,dx = \left/ \begin{matrix} t = 1+5x^2 \\ \frac{dt}{dx} = 10x \\ dt = 10x\,dx \end{matrix} \right/ =$$

$$= \int t^3 \frac{dt}{10} = \frac{1}{10}\int t^3\,dt = \frac{1}{10}\cdot\frac{t^4}{4} + C =$$

$$= \frac{1}{40}(1+5x^2)^4 + C$$

Glöm inte kontroll.

Läs in listan på sid 239.

Ex. $$\int (e^{4x} - \sin 2x)\,dx = \frac{e^{4x}}{4} + \frac{\cos 2x}{2} + C$$

Ex. $$\int \frac{1}{1+4x}\,dx = \frac{\ln|1+4x|}{4} + C$$

Ex. $\displaystyle\int \frac{x}{1+4x^2}\,dx = \left/ \begin{array}{l} t = 1+4x^2 \\ \frac{dt}{dx} = 8x \\ \frac{dt}{8} = x\,dx \end{array} \right/ =$

$$= \int \frac{\frac{dt}{8}}{t} = \frac{1}{8}\int \frac{1}{t}\,dt =$$

$$= \frac{1}{8}\ln|t| + C = \frac{1}{8}\ln(1+4x^2) + C$$

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trick. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

Bra att kunna.

Ex. $\displaystyle\int \frac{x}{1+4x^2}\,dx = \frac{1}{8}\int \frac{8x}{1+4x^2}\,dx =$

$$= \frac{1}{8}\ln(1+4x^2) + C$$

Partiell integration

$f = f(x)$
$g = g(x)$

$$\boxed{\int f \cdot g \, dx = Fg - \int F \cdot g' \, dx}$$

Bevis: $\frac{d}{dx}(F \cdot g) = fg + Fg'$

$$\int \frac{d}{dx}(F \cdot g) = \int f \cdot g + \int Fg'$$

$$Fg = \int fg + \int Fg'$$

$$\therefore \int fg = Fg - \int Fg'$$

Ex.

$$\int \underset{\downarrow}{x} \underset{\uparrow}{e^{2x}} \, dx \underset{p.i}{=} \frac{e^{2x}}{2} x - \int \frac{e^{2x}}{2} \cdot 1 \, dx =$$

$f = e^{2x}$
$g = x$

$$= \frac{x}{2} e^{2x} - \frac{1}{2} \int e^{2x} \, dx =$$

$$= \frac{x}{2} e^{2x} - \frac{1}{2} \frac{e^{2x}}{2} + C =$$

$$= \left(\frac{x}{2} - \frac{1}{4} \right) e^{2x} + C$$

Gör kontroll.

Ex.

$$\int \underset{\downarrow}{x^2} \underset{\uparrow}{\sin 5x}\, dx \underset{p.i}{=} -\frac{\cos 5x}{5} x^2 - \int -\frac{\cos 5x}{5} 2x\, dx =$$

$$= -\frac{x^2}{5}\cos 5x + \frac{2}{5}\int \underset{\downarrow}{x}\, \underset{\uparrow}{\cos 5x}\, dx \underset{\text{p.i igen.}}{=}$$

$$= -\frac{x^2}{5}\cos 5x + \frac{2}{5}\left(\frac{\sin 5x}{5} x - \int \frac{\sin 5x}{5}\cdot 1\, dx\right) =$$

obs! Jätteviktig parentes.

$$= -\frac{x^2}{5}\cos 5x + \frac{2x}{25}\sin 5x - \frac{2}{25}\int \sin 5x\, dx =$$

$$= -\frac{x^2}{5}\cos 5x + \frac{2x}{25}\sin 5x + \frac{2}{125}\cos 5x + C$$

Ex.

$$\int \underset{\uparrow}{x^2} \underset{\downarrow}{\ln x}\, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}\, dx =$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

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trick. Skjut in en "etta".

Vanligt.

Ex. $\int \ln x \, dx = \int 1 \cdot \ln x \, dx = \int \underset{\uparrow}{1} \cdot \underset{\downarrow}{\ln x} \, dx =$

$\underset{p.i}{=} x \ln x - \int x \frac{1}{x} dx = x \ln x - \int 1 \, dx =$

$= x \ln x - x + C$

Ex. $\int \arctan x \, dx = \int \underset{\uparrow}{1} \cdot \underset{\downarrow}{\arctan x} \, dx \underset{p.i}{=}$

$= x \arctan x - \int x \cdot \frac{1}{1+x^2} dx =$

$= x \arctan x - \int \frac{x}{1+x^2} dx =$

$= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} dx =$

$= x \arctan x - \frac{1}{2} \ln |1+x^2| + C$

Blandade övningar.

Ex. $$\int 4xe^{x^2}dx = \left/ \begin{array}{l} t = x^2 \\ \frac{dt}{dx} = 2x \\ dt = 2x\,dx \end{array} \right/ =$$

$$= \int 2e^t dt = 2e^t + C = 2e^{x^2} + C$$

Obs! Vi själva har facit.

Ex. $$\int x^3 e^{2x} dx = \left/ \begin{array}{l} \text{p.i} \\ \text{3 gånger} \end{array} \right/ =$$

$$= \int \underset{\downarrow}{x^3} \underset{\uparrow}{e^{2x}} dx = \frac{e^{2x}}{2}x^3 - \int \frac{e^{2x}}{2} 3x^2 dx =$$

$$= \frac{x^3}{2}e^{2x} - \frac{3}{2}\int \underset{\downarrow}{x^2} \underset{\uparrow}{e^{2x}} dx =$$ p.i igen

$$= \frac{x^3}{2}e^{2x} - \frac{3}{2}\left(\frac{e^{2x}}{2}x^2 - \int \frac{e^{2x}}{2} 2x\, dx \right) =$$

parentes

$$= \frac{x^3}{2}e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{3}{2}\left(\frac{e^{2x}}{2}x - \frac{e^{2x}}{4} \right) + C$$

Ex. $\int \sin\sqrt{x}\,dx = \left/ \begin{array}{l} t=\sqrt{x} \\ x=t^2 \\ \frac{dx}{dt}=2t \\ dx = 2t\,dt \end{array} \right/ =$

$= \int (\sin t)2t\,dt = 2\int \underset{\downarrow}{t}\, \underset{\uparrow}{\sin t}\,dt \underset{p.i}{=}$

$= 2\left((-\cos t)t - \int(-\cos t)\cdot 1\,dt\right) =$

$= -2t\cos t + 2\sin t + C =$

$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

Ex. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx =$

$= -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C$

Ex. $\displaystyle\int \frac{x^4}{x^2+1}\,dx = \left/ \begin{array}{c} \text{Polynomdiv.} \\ \dfrac{x^2(x^2+1)-(x^2+1)+1}{x^2+1} \end{array} \right/$

$$= \int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx =$$

$$= \frac{x^3}{3} - x + \arctan x + C$$

Ex. $\displaystyle\int \frac{7}{x^2+9}\,dx = \frac{7}{9}\int \frac{1}{\frac{x^2}{9}+1}\,dx =$

$$= \frac{7}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx = \left/ \begin{array}{c} t = \frac{x}{3} \\ \frac{dt}{dx} = \frac{1}{3} \\ dx = 3dt \end{array} \right/ =$$

$$= \frac{7}{9}\int \frac{3dt}{t^2+1} = \frac{7}{3}\arctan t + C =$$

$$= \frac{7}{3}\arctan\left(\frac{x}{3}\right) + C$$
