Fö2 | Läs 5.3 – 5.5.

Rationella uttryck.

$$\int \frac{f(x)}{g(x)}\,dx$$ där f, g polynom.

① Om grad $f \geq$ grad g utför polynomdivision.

$$\frac{f(x)}{g(x)} = \text{kvot} + \frac{\text{rest}}{g(x)}$$

② Om grad(tälj) < grad(nämn): utför ev. partialbråksuppdelning (PBU).

a) Faktorisera först.

b) Gör ansats.

c) Bestäm konstanter.
Gärna med handpåläggning.

$$\text{Ex. } \int \frac{2x-1}{x^2+2x-3}\,dx = \underset{\text{Se nedan.}}{\text{PBU}^{*}}$$

$$= \int \left(\frac{\frac{1}{4}}{x-1} + \frac{\frac{7}{4}}{x+3} \right) dx = \frac{1}{4}\int \frac{1}{x-1}\,dx + \frac{7}{4}\int \frac{1}{x+3}\,dx$$

$$= \frac{1}{4}\ln|x-1| + \frac{7}{4}\ln|x+3| + C$$

$$* \quad \frac{2x-1}{x^2+2x-3} = \frac{2x-1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} =$$

$$= \frac{A(x+3) + B(x-1)}{(x-1)(x+3)} = \frac{(A+B)x + 3A - B}{(x-1)(x+3)}$$

Jämför koefficienter framför x, x^0.

ger $\begin{cases} A + B = 2 \\ 3A - B = -1 \end{cases} \iff \begin{cases} A = 1/4 \\ B = 7/4 \end{cases}$

Snabbare med .

$$\frac{2x-1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

A söks. mult. med $(x-1)$ låt $x \to 1$.

B söks. mult med $(x+3)$ låt $x \to -3$

Korrekt ansats.

Ex. $$\frac{5x+7}{x(x-1)^3(x^2+1)} =$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{x^2+1}$$

Ex. Partialbråksuppdela $\frac{1}{x^3-x}$

L: $$\frac{1}{x^3-x} = \frac{1}{x(x^2-1)} = \frac{1}{x(x+1)(x-1)} =$$

$$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$\Rightarrow$ $A = -1$, $B = \frac{1}{2}$, $C = \frac{1}{2}$

Ex. $$\frac{5x-1}{x^3-x^2+x-1} \overset{\text{PBU.}}{=} \frac{5x-1}{(x-1)(x^2+1)} =$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(x-1)(Bx+C)}{(x-1)(x^2+1)}$$

$\Rightarrow$ $A = 2$

koeff. x^2: $A+B=0$ $\quad B=-2$

x: $C-B=5$ $\quad C=3$

x^0: $A-C=-1$

Olika slutlägen

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$

Ex. $$\int \frac{1}{2-x}\,dx = \frac{\ln|2-x|}{-1} + C$$

Ex. $$\int \frac{1}{(x+1)^2}\,dx = \int (x+1)^{-2}\,dx =$$

$$= \frac{(x+1)^{-1}}{-1} + C = \frac{-1}{x+1} + C$$

Ex. $$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx =$$

$$= \frac{1}{4}\,\frac{\arctan\left(\frac{x}{2}\right)}{\frac{1}{2}} + C = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

Kompensera för inre derivatan.

O.K. inre derivatan = konstant.

Ex. $\displaystyle\int \frac{x+1}{x^2+2x+5}\,dx =$ ↑ leta efter

Viktig! $\boxed{\displaystyle\int \frac{f'}{f}\,dx = \ln|f| + C}$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx = \frac{1}{2}\ln(x^2+2x+5) + C$$

Ex. $\displaystyle\int \frac{x+3}{x^2+4x+5}\,dx = \int \frac{x+2+1}{(x+2)^2+1}\,dx =$

$$= \left/ \begin{aligned} t &= x+2 \\ \frac{dt}{dx} &= 1 \\ dt &= dx \end{aligned} \right/ = \int \frac{t+1}{t^2+1}\,dt =$$

$$= \int \frac{t}{t^2+1}\,dt + \int \frac{1}{t^2+1}\,dt =$$

$$= \frac{1}{2}\int \frac{2t}{t^2+1}\,dt + \int \frac{1}{t^2+1}\,dt =$$

$$= \frac{1}{2}\ln(t^2+1) + \arctan t + C$$ där $t = x+2$.

Trigonometriska uttryck.

$$\int f(\sin x)\cos x\,dx = \left/ \begin{array}{l} \text{Sub.} \\ t=\sin x \\ dt=\cos x\,dx \end{array} \right/ = \ldots$$

$$\int f(\cos x)\sin x\,dx = \left/ \begin{array}{l} t=\cos x \\ \text{osv.} \end{array} \right/$$

Ex. $$\int \sin^5 x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx =$$

$$= \left/ \begin{array}{l} t=\cos x \\ \frac{dt}{dx} = -\sin x \end{array} \right/ = \int (1-t^2)^2(-1)\,dt =$$

$$= \int (-t^4+2t^2-1)\,dt = -\frac{t^5}{5}+\frac{2t^3}{3}-t+C =$$

$$= -\frac{\cos^5 x}{5}+\frac{2}{3}\cos^3 x-\cos x + C$$

Ex. $\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx =$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C =$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

Kom ihåg!

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$$

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2}}$$

Rep. även Eulers formler.

Allmänt: $\int \sin^n x \, dx$, $\int \cos^n x \, dx$ n heltal $n \geq 1$.

n udda. Bryt ut. Nyttja trig. ettan

n jämn. Formler se ovan.

Ex. $\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx =$

$$= \int \frac{1+2\cos 2x+\cos^2 2x}{4}\,dx =$$

$$= \frac{1}{4}\int\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx =$$

$$= \frac{1}{4}\int\left(\frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x\right)dx =$$

$$= \frac{1}{4}\left(\frac{3}{2}x+\frac{2\sin 2x}{2}+\frac{1}{2}\cdot\frac{\sin 4x}{4}\right)+C =$$

$$= \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

Ex. $\int \frac{1}{\sin x}\,dx = \int \frac{\sin x}{\sin^2 x}\,dx =$

$$= \int \frac{\sin x}{1-\cos^2 x}\,dx = \left|\begin{array}{l} t=\cos x \\ \frac{dt}{dx}=-\sin x \\ (-1)dt=\sin x\,dx \end{array}\right| =$$

$$= \int \frac{(-1)\,dt}{1-t^2} = \int \frac{1}{t^2-1}\,dt =$$

$$= \int \frac{1}{(t+1)(t-1)}\,dt = \int \left(\frac{A}{t+1} + \frac{B}{t-1} \right) dt$$

PBU

ger $A = -\frac{1}{2}$

$B = \frac{1}{2}$

$$= \int \left(\frac{-\frac{1}{2}}{t+1} + \frac{\frac{1}{2}}{t-1} \right) dt =$$

$$= -\frac{1}{2}\ln|t+1| + \frac{1}{2}\ln|t-1| + C =$$

$$= \frac{1}{2}\left(\ln|t-1| - \ln|t+1| \right) + C =$$

$$= \frac{1}{2}\ln\left| \frac{t-1}{t+1} \right| + C =$$

$$= \frac{1}{2}\ln\left| \frac{\cos x - 1}{\cos x + 1} \right| + C =$$

$$= \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x} \right) + C$$

om inget annat går.

Sub. $t = \tan\frac{x}{2}$

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2t}{t^2+1}$$

Bryt ut $\cos^2\frac{x}{2}$.

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{1-t^2}{t^2+1}$$

$$x = 2\arctan t$$

$$\frac{dx}{dt} = 2\frac{1}{1+t^2} \iff dx = \frac{2}{1+t^2}\,dt$$

Ex. $\displaystyle\int \frac{1}{\sin x}\,dx = \left/ \begin{matrix} t=\tan\frac{x}{2} \\ \text{osv} \end{matrix} \right/ =$

$$= \int \frac{1}{\frac{2t}{t^2+1}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{1}{t}\,dt = \ln|t| + C =$$

$$= \ln\left|\tan\frac{x}{2}\right| + C$$

Rotuttryck

Se sid 239.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{\sqrt{x^2+\alpha}}\,dx = \ln\left|x+\sqrt{x^2+\alpha}\right| + C$$

$\boxed{\text{Sätt } t = \text{hela rotuttrycket}}$.

Ex. $$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \left/ \begin{array}{c} t=\sqrt{x^2+1} \\ x^2 = t^2-1 \\ 2x\,dx = 2t\,dt \end{array} \right/ =$$

$$= \int \frac{(t^2-1)\,t\,dt}{t} = \int (t^2-1)\,dt =$$

$$= \frac{t^3}{3} - t + C = \frac{(x^2+1)^{3/2}}{3} - \sqrt{x^2+1} + C$$

Kontroll.

$$\frac{1}{2}(x^2+1)^{1/2}\cdot 2x - \frac{1}{2\sqrt{x^2+1}}\cdot 2x = \frac{x\sqrt{x^2+1}\sqrt{x^2+1} - x}{\sqrt{x^2+1}} =$$

$$= \frac{x^3+x-x}{\sqrt{x^2+1}} = \frac{x^3}{\sqrt{x^2+1}}$$

$$-\pi/2 \le t \le \pi/2$$

Ex. $$\int \sqrt{1-x^2}\,dx = \left/ \begin{array}{c} x = \sin t \\ \dfrac{dx}{dt} = \cos t \\ dx = \cos t\,dt \end{array} \right/ =$$

$$= \int \sqrt{1-\sin^2 t}\,\cos t\,dt = \int \sqrt{\cos^2 t}\,\cos t\,dt =$$

$$= \int |\cos t|\cos t\,dt = \int \cos t \cos t\,dt =$$

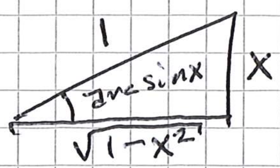
$$= \int \cos^2 t\,dt = \int \left(\frac{1+\cos 2t}{2}\right) dt =$$

$$= \frac{1}{2} t + \frac{1}{2}\cdot\frac{\sin 2t}{2} + C =$$

$$= \frac{1}{2}\arcsin x + \frac{1}{4}\sin 2\arcsin x + C =$$

$$= \frac{1}{2}\arcsin x + \frac{1}{4}\, 2 \sin\arcsin x \cos\arcsin x + C$$

$$= \frac{1}{2}\arcsin x + \frac{1}{2} x\sqrt{1-x^2} + C$$



jfr med ex 5.40