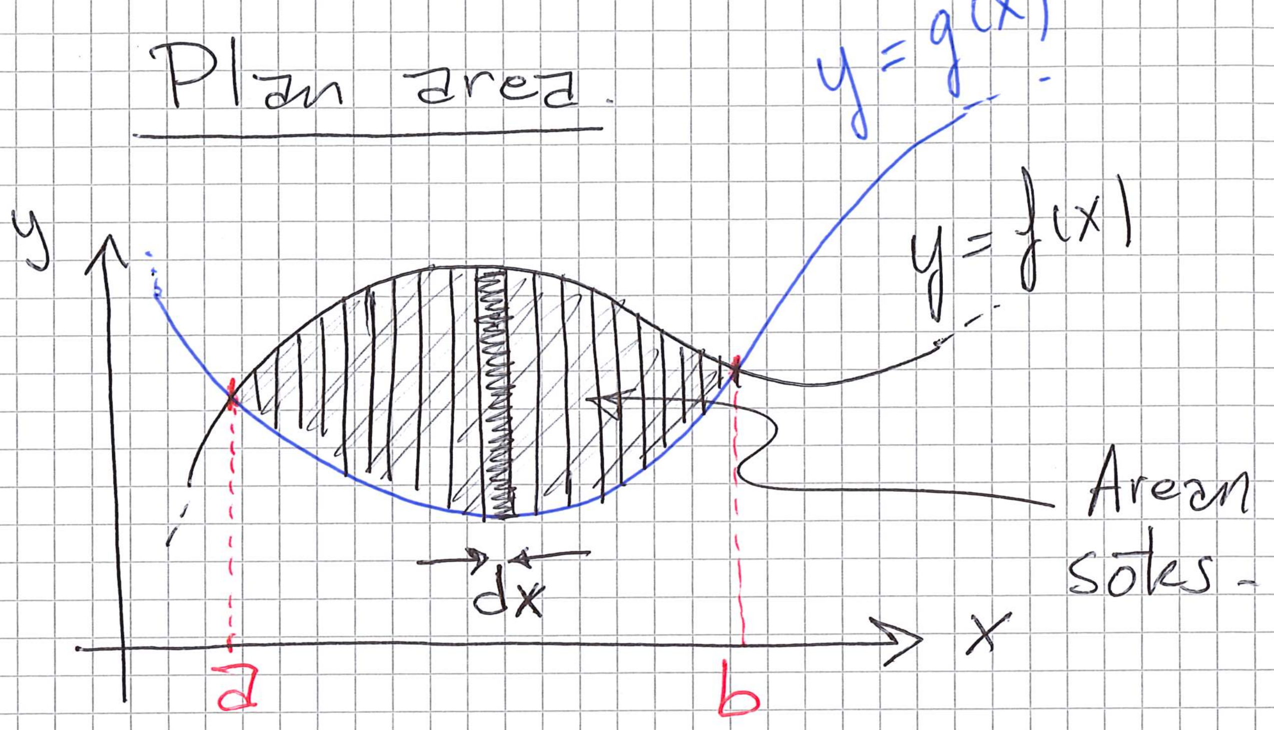
Fö 4 Läs 7.1 − 7.3.

Plan area.



$f(x) = g(x)$ ger a och b.

Dela intervallet $[a, b]$ i smala strimlor.

bredd $= dx$

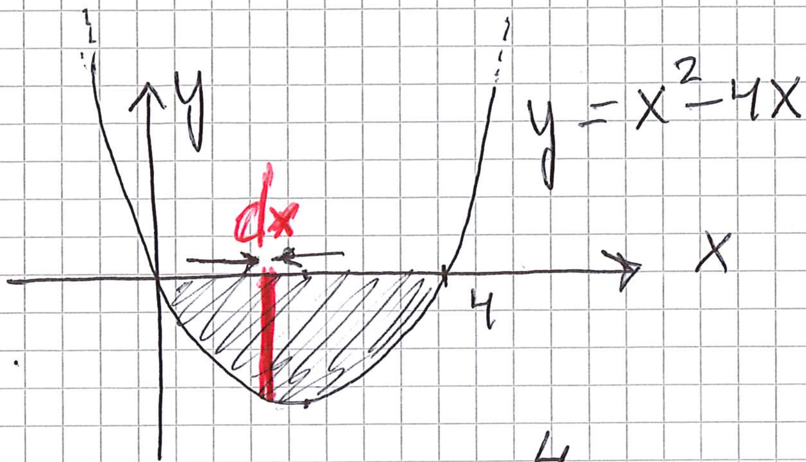
höjden $=$ (övre − undre) $= f(x) - g(x)$

$$\text{Arean} = \int_a^b \underbrace{(f(x) - g(x))\,dx}_{= dA}$$

Läs $\int$-tecknet som en $\sum$ (Summering).

Ex. Beräkna <u>arean</u> av det begränsade området mellan $y = x^2 - 4x$ och x-axeln.

L: <u>Rita</u>.



x-axeln "heter" $y = 0$.

$$A = \int_0^4 \left(0 - (x^2 - 4x)\right) dx = \int_0^4 (4x - x^2)\, dx =$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} - 0 = \frac{32}{3}$$

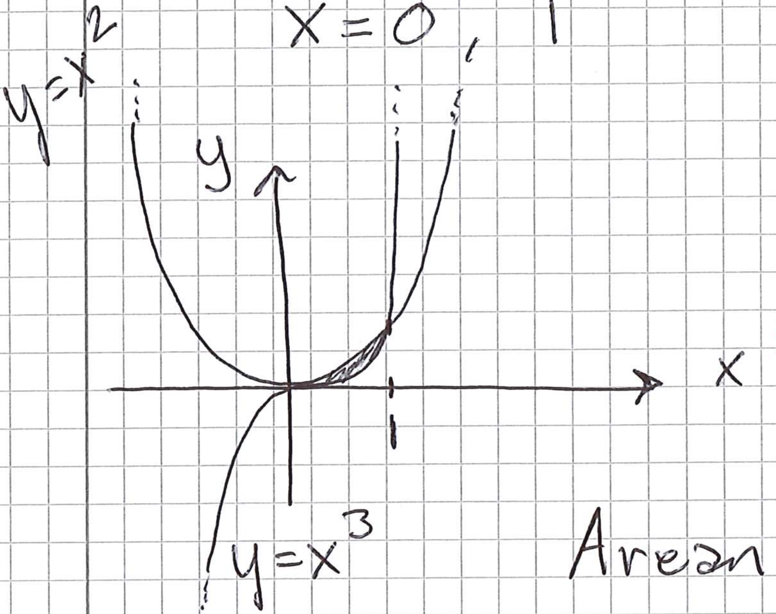
Svar: $\frac{32}{3}$ a.e.

Ex. Beräkna arean av det begränsade området som ligger mellan kurvorna $y = x^2$ och $y = x^3$.

L. $y = x^2$, $y = x^3$

$x^2 = x^3 \iff 0 = x^2(x-1) \iff$

$x = 0 , 1$



Sätt in värde mellan 0 och 1.

Ger övre resp. undre.

$$\text{Arean} = \int_0^1 \underbrace{(x^2 - x^3)}_{\text{höjd}} \cdot \underbrace{dx}_{\text{bredd}} =$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \underline{\underline{\frac{1}{12}}} \text{ a.e.}$$

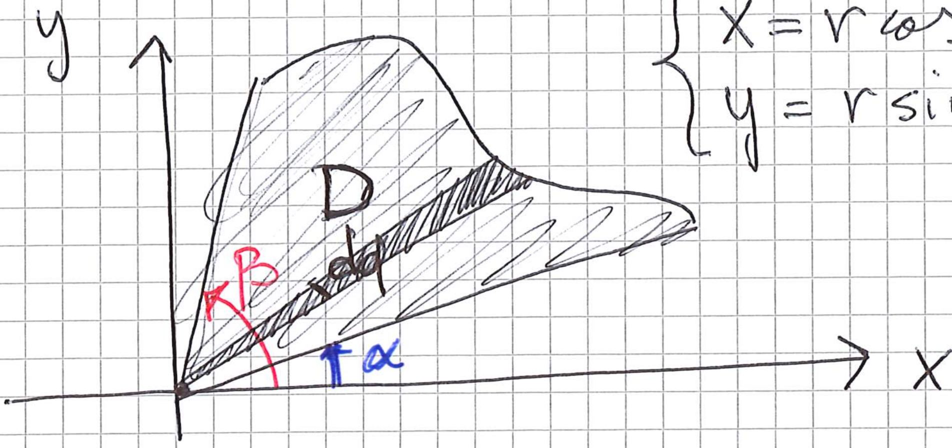
<u>Exempel 7-2.</u>

Boken sid 314.

L: Rita ... osv.

Bra start.

Plan area i polära koordinater



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$D = \{(x,y) : \alpha \leq \varphi \leq \beta,\ 0 \leq r \leq r(\varphi)\}$$

$$dA = \pi r^2 \cdot \frac{d\varphi}{2\pi}$$ = arean av en cirkelsektor.

Cirkelns area. andelen av en cirkel.

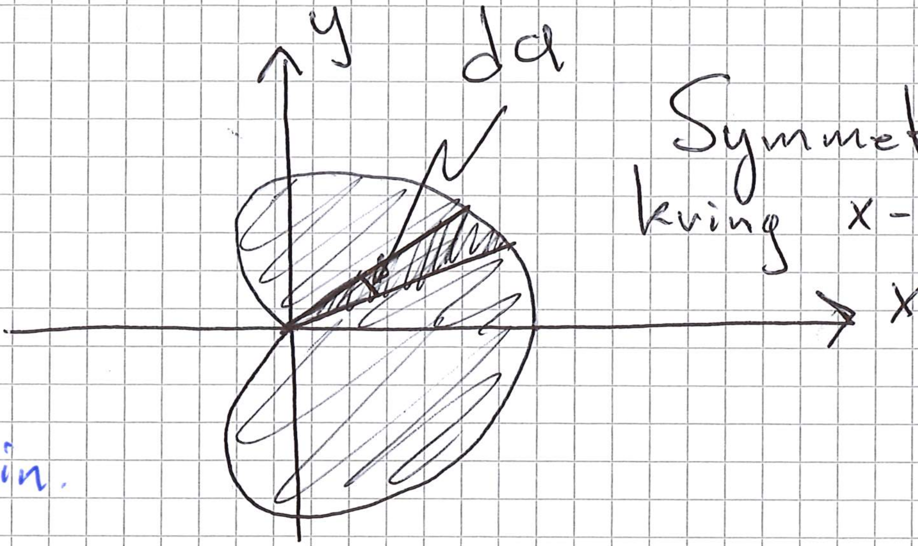
$$\text{Arean} = \int_{\alpha}^{\beta} \frac{r^2(\varphi)}{2}\, d\varphi$$

Ex. $r = 1 + \cos\varphi$, $-\pi \leq \varphi \leq \pi$

Bestäm arean innanför.

L: α = riktningen.

$r(\alpha)$ = avståndet ut från origo.



Symmetri kring x-axeln.

$$r = 1 + \cos\alpha \quad , \quad -\pi \le \alpha \le \pi .$$

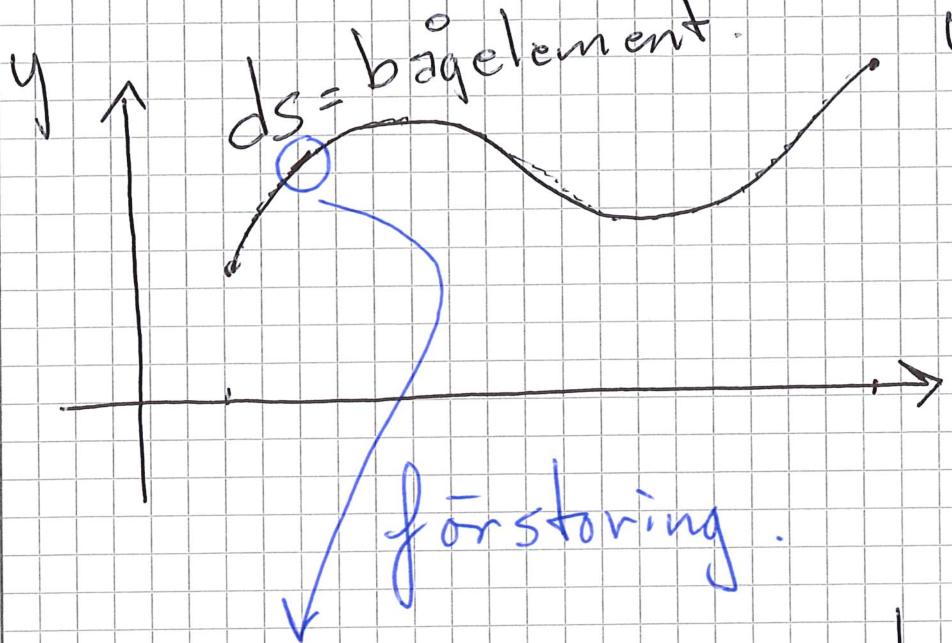
Symmetrin. Halva intervallet $\cdot 2$.

$$A = 2\int_0^{\pi} \underbrace{\pi r^2 \frac{d\alpha}{2\pi}}_{\text{smal cirkelsektors area.}} = \int_0^{\pi} (1+\cos\alpha)^2 d\alpha =$$

$\int$ är summering.

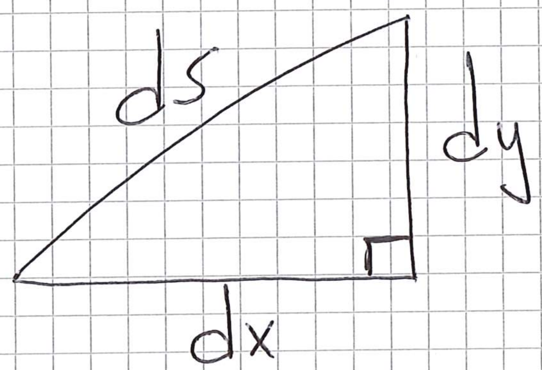
$$= \int_0^{\pi} \Big(1 + 2\cos\alpha + \underbrace{\cos^2\alpha}_{=\frac{1+\cos 2\alpha}{2}}\Big) d\alpha = \ldots = \underline{\underline{\frac{3\pi}{2} \text{ a.e}}}$$

Kurvlängd



$y = f(x)$,
$a \leq x \leq b$

f' kont.



$$ds = \sqrt{dx^2 + dy^2}$$

Summera
små bågelement

$$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{1 + (y'(x))^2}\, dx$$

Ex. Bestäm längden av kurvan

$$y = \frac{x^2}{4} - \frac{\ln x}{2}, \quad 1 \leq x \leq 4.$$

$$L = \int \sqrt{dx^2 + dy^2} = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx =$$

$$= \int_1^4 \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, dx =$$

$$= \int_1^4 \sqrt{1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}}\, dx =$$

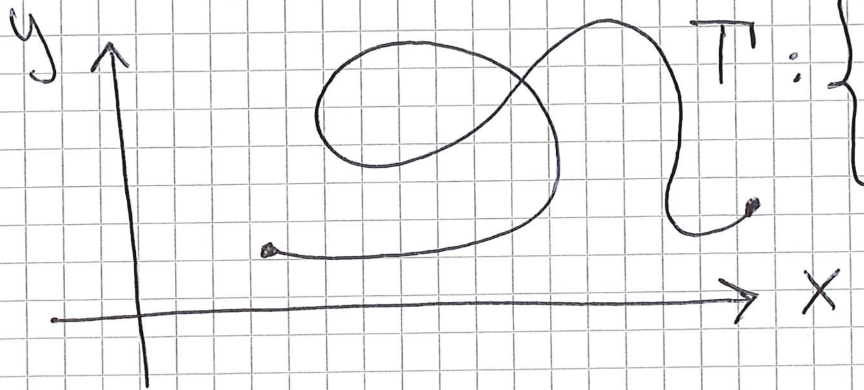
$$= \int_1^4 \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}}\, dx = \int_1^4 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^4 \left|\frac{x}{2} + \frac{1}{2x}\right| dx = \int_1^4 \left(\frac{x}{2} + \frac{1}{2x}\right) dx =$$

$\sqrt{x^2} = |x|$

$$= \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^4 = \frac{15}{4} + \ln 2 \text{ l.e}$$

Parametrisering



$$T: \begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad t_0 \le t \le t_1$$

t = tiden.

$(x(t), y(t))$ = partikelns läge.

Sök kurvan T:s längd, L

$$L = \int ds = \int \sqrt{dx^2 + dy^2} =$$

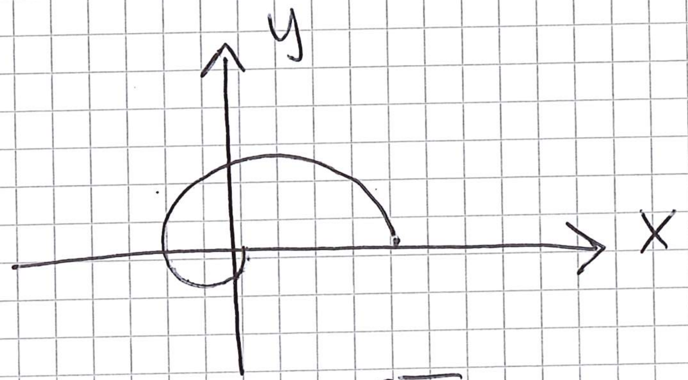
$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$$

$$= \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Ex. Beräkna längden av kurvan

$$\begin{cases} x = e^{-t}\cos t \\ y = e^{-t}\sin t \end{cases}, \quad 0 \le t \le 2\pi.$$

L.



$$L = \int \sqrt{dx^2 + dy^2} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$$

$$= \left/ \begin{array}{l} \dfrac{dx}{dt} = -e^{-t}\cos t - e^{-t}\sin t \\ \dfrac{dy}{dt} = -e^{-t}\sin t + e^{-t}\cos t \end{array} \right/ =$$

$$= \int_0^{2\pi} \sqrt{\left(-e^{-t}(\cos t + \sin t)\right)^2 + \left(e^{-t}(\cos t - \sin t)\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{e^{-2t}\left(1 + \cancel{2\sin t\cos t} + 1 - \cancel{2\sin t\cos t}\right)}\, dt$$

$$= \int_{-\pi}^{\pi} \sqrt{r^2 + (r')^2}\, d\varphi = \left/ \begin{matrix} r = 1 + \cos\varphi \\ r' = -\sin\varphi \end{matrix} \right/ =$$

$$= 2 \int_{0}^{\pi} \sqrt{1 + 2\cos\varphi + \cos^2\varphi + \sin^2\varphi}\, d\varphi =$$

trig. etta.

Symmetrin

$$= 2 \int_{0}^{\pi} \sqrt{2 + 2\cos\varphi}\, d\varphi =$$

$$= 2\sqrt{2} \int_{0}^{\pi} \sqrt{1 + \cos\varphi}\, d\varphi =$$

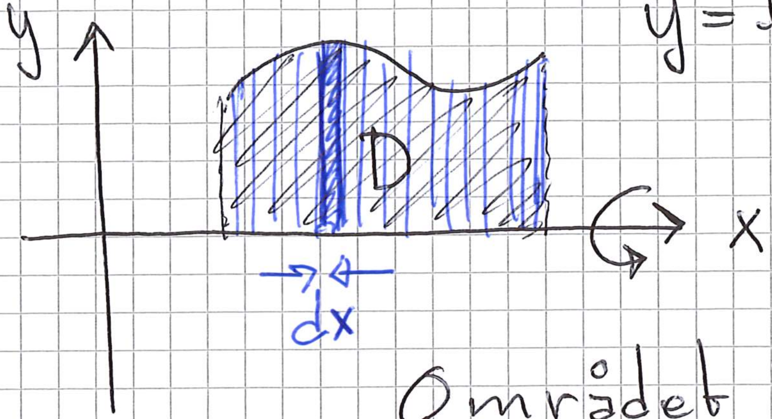
$$= 2\sqrt{2} \int_{0}^{\pi} \sqrt{2\cos^2\frac{\varphi}{2}}\, d\varphi = 4 \int_{0}^{\pi} \left|\cos\frac{\varphi}{2}\right| d\varphi$$

$$= 4 \int_{0}^{\pi} \cos\frac{\varphi}{2}\, d\varphi = 4 \left[\frac{\sin\frac{\varphi}{2}}{\frac{1}{2}} \right]_{0}^{\pi} =$$

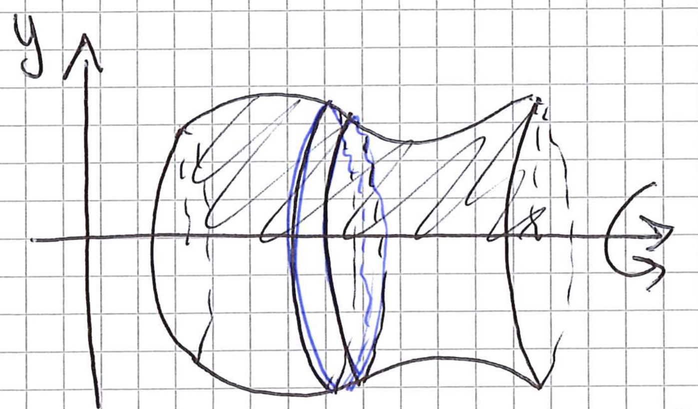
$$= 8 \left(\sin\frac{\pi}{2} - \sin 0 \right) = \underline{\underline{8 \text{ l.e.}}}$$

Rotationsvolym, V

$y = f(x)$, $a \leq x \leq b$



Området D roteras ett varv kring x-axeln.



Uppkommer en massiv kropp.

Cirkelskivas area $= \pi r^2$

$$V = \int_a^b \pi y^2 \, dx$$

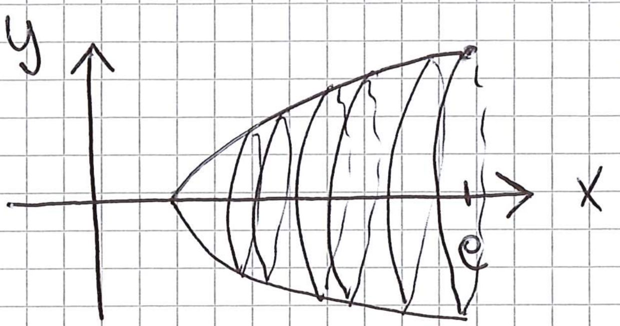
"Skivformeln".

y = radien

dx = tjockleken.

Ex. Området mellan x-axeln och kurvan $y = \ln x$, $1 \le x \le e$ roteras ett varv kring <u>x-axeln</u>. Bestäm den uppkomna kroppens volym.

L:



$$V = \int_1^e \pi y^2 \, dx = \pi \int_1^e (\ln x)^2 \, dx =$$

$$\pi \int_1^e 1 \, (\ln x)^2 \, dx \underset{p.i}{=}$$

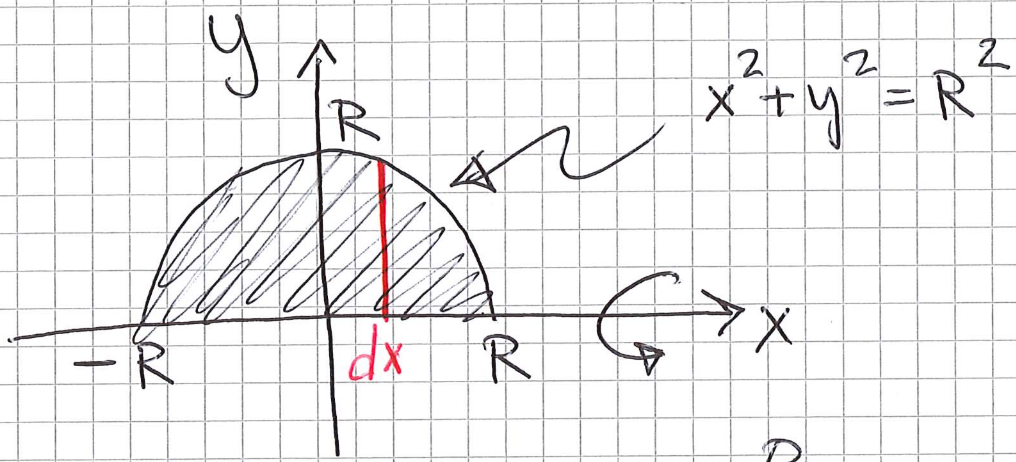
$$= \pi \left(\left[x (\ln x)^2 \right]_1^e - \int_1^e x \, 2 \ln x \cdot \frac{1}{x} \, dx \right) =$$

$$= \pi e - 2\pi \int_1^e \ln x \, dx = \pi e - 2\pi \left[x \ln x - x \right]_1^e =$$

skjut in etta. osv.

$$= \pi e - 2\pi(+1) = \underline{\underline{\pi(e-2)}} \text{ v.e.}$$

Ex. Klotets volym



$$V=\int_{-R}^{R}\pi y^2\,dx = 2\pi\int_{0}^{R}(R^2-x^2)\,dx =$$

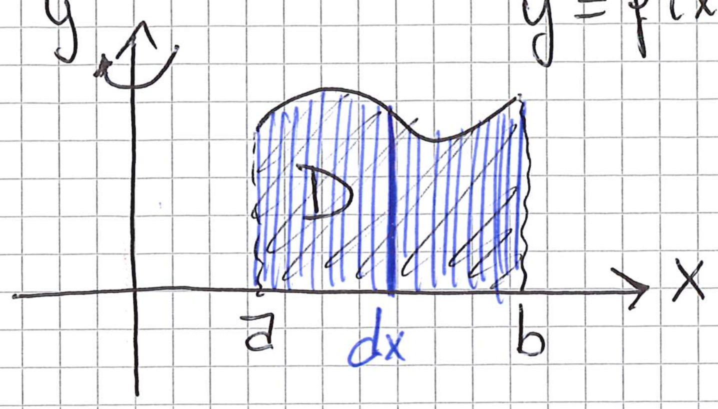
$$= 2\pi\left[R^2x-\frac{x^3}{3}\right]_0^R =$$

$$= 2\pi\left(R^3-\frac{R^3}{3}-0\right)=\frac{4\pi R^3}{3}$$

$$\text{Klotets volym} = \frac{4\pi R^3}{3}$$

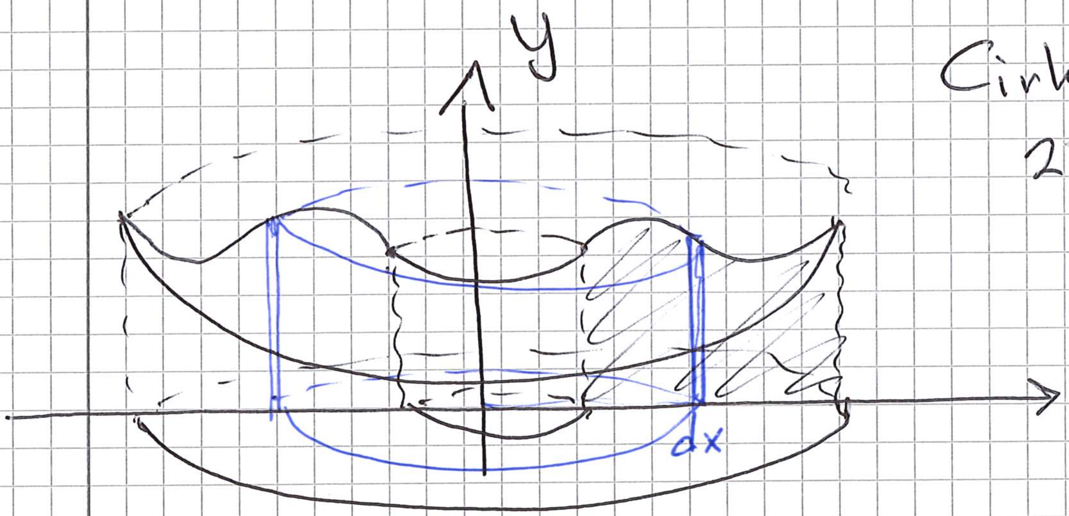
Rörformeln

$y = f(x)$, $a \le x \le b$



D (område) roteras ett varv kring y-axeln.

Bestäm den uppkomna kroppens volym.



Cirkels omkrets
$2\pi r$

$$V = \int_a^b 2\pi x\, y\, dx$$

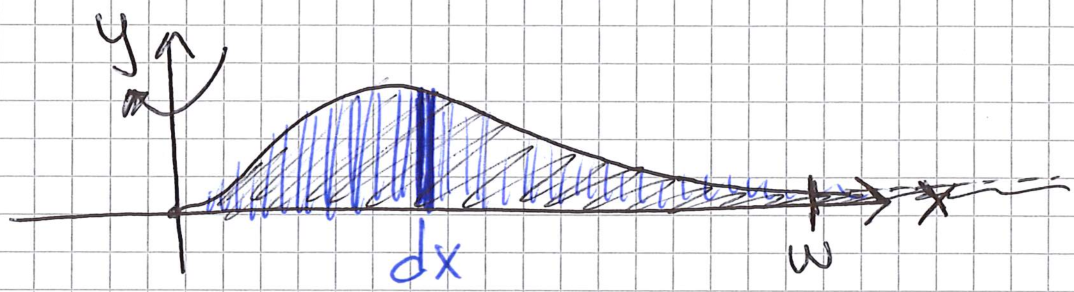
x = radien

y = rörets höjd.

dx = tjockleken.

Ex. Beräkna volymen av den kropp som uppstår då området $0 \leq y \leq x^2 e^{-x^2}$, $x \geq 0$ roteras ett varv kring y-axeln.

L:



$$V = \int_0^{\infty} 2\pi x y \, dx$$

$$\int_0^{\omega} 2\pi x x^2 e^{-x^2} dx = \left/ \begin{array}{l} t = x^2 \\ dt = 2x\,dx \\ \text{byt gränser} \end{array} \right/ =$$

$$= \int_0^{\omega^2} \pi \underset{\downarrow}{t} \underset{\uparrow}{e^{-t}} dt \underset{p.i}{=} \pi \left(\left[-e^{-t} t \right]_0^{\omega^2} - \int -e^{-t} dt \right) =$$

$$= \pi \left(-\frac{\omega^2}{e^{\omega^2}} + 0 + \left[\frac{e^{-t}}{-1} \right]_0^{\omega^2} \right) =$$

$$= \pi \left(-\underbrace{\frac{\omega^2}{e^{\omega^2}}}_{\to 0} - \underbrace{\frac{1}{e^{\omega^2}}}_{\to 0} + 1 \right) \to \pi \text{ då } \omega \to \infty$$