Fö5 | Läs 7.3–7.4

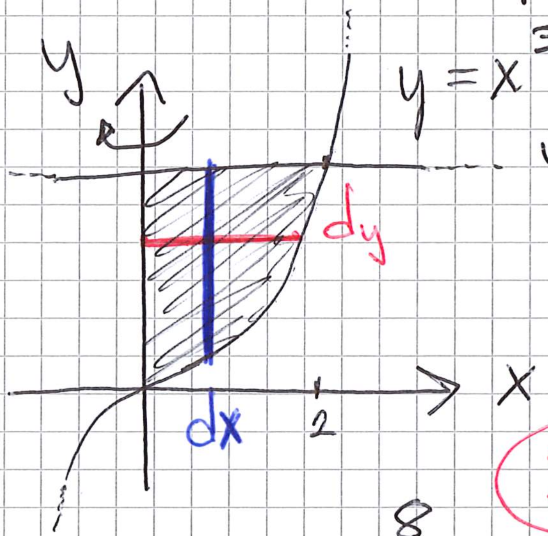
Mer om volymer.

Ex. Området i 1:a kvadranten som begränsas av $y=x^3$, $y=8$ samt y-axeln roteras ett varv kring y-axeln.

Beräkna kroppens volym.

L:

$y=x^3 \Leftrightarrow x=\sqrt[3]{y}$



Skivformeln

$$V=\int_0^8 \pi x^2 dy = \pi\int_0^8 (\sqrt[3]{y})^2 dy = \pi\left[\frac{y^{5/3}}{\frac{5}{3}}\right]_0^8 =$$

$$= \pi\frac{3}{5}\left(8^{5/3}-0\right) = \frac{3\pi}{5}(2^3)^{\frac{5}{3}} = \frac{96\pi}{5}.$$

Alt.

$$V=\int_0^2 2\pi x \underbrace{(8-x^3)}_{=\text{höjden}} dx = \ldots = \frac{96\pi}{5}$$

Rörformeln

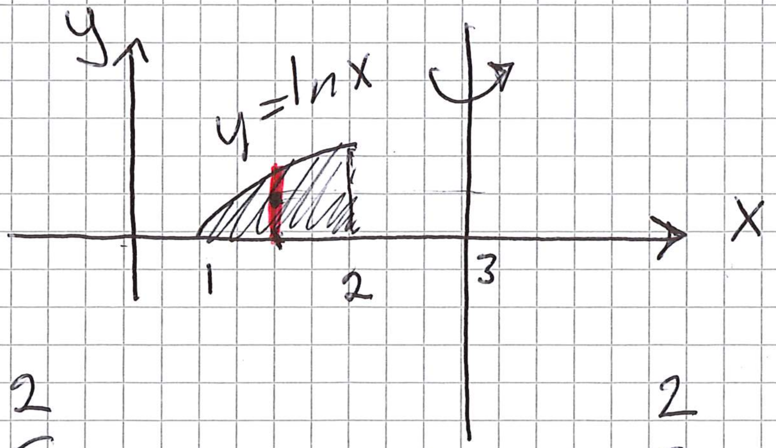
Pappos-Guldins regel.

(I) Volymen = (TP:s väg) · arean.

TP = tyngdpunkt.

Ex. Området $D = \{(x,y) : 0 \leq y \leq \ln x, 1 \leq x \leq 2\}$ får rotera ett varv kring linjen $\underline{x = 3}$. Beräkna volymen av den kropp som uppkommer.

L:



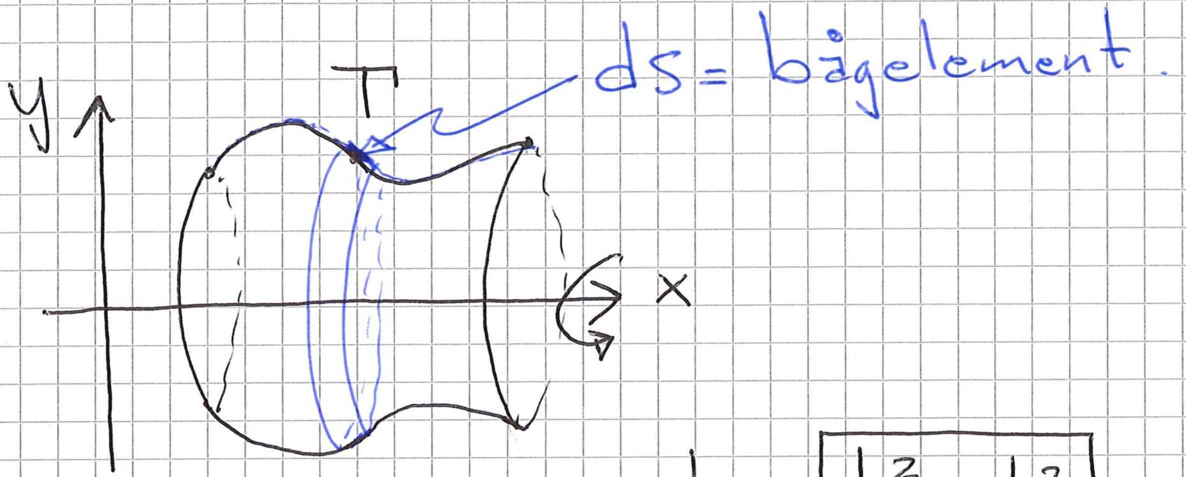
$$V = \int_1^2 (\text{TP:s väg})\, \text{arean} = \int_1^2 2\pi(3-x)\, y\, dx =$$

$$= 2\pi \int_1^2 (3-x) \ln x \, dx = \ldots = 2\pi\left(4\ln 2 - \frac{9}{4}\right)$$

Se exempel 7.11. Sid 327.

Rotationsarea.

Låt kurvan Γ rotera ett varv runt x-axeln. Hur stor blir rotationsarean?



$$ds = \sqrt{dx^2 + dy^2}$$

$$A = \int 2\pi y \, ds$$

Pappos-Guldins regel nr (II)

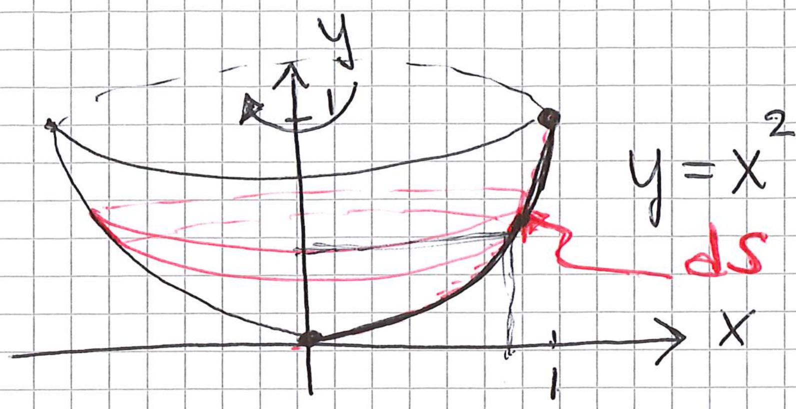
Rotationsarean = (TP:s väg) · kurvlängden

Ex. Rotera kurvan $y = x^2$, $0 \leq x \leq 1$ ett varv kring y-axeln.

Då fås en paraboloid.

Bestäm rotationsarean.

L:



$y = x^2, \quad 0 \le x \le 1$

$$A = \int (\text{TP:s väg})\, ds = \int 2\pi x \, ds =$$

$\underline{\underline{x = \text{radien}}}$.

$$= \int 2\pi x \sqrt{dx^2 + dy^2} = \int 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx =$$

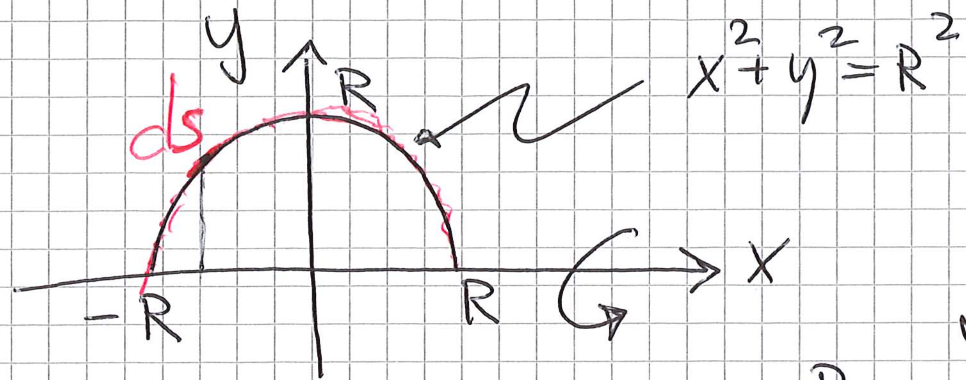
$$= 2\pi \int_0^1 x \sqrt{1 + (y'(x))^2}\, dx = \left/ \begin{array}{l} y(x) = x^2 \\ y'(x) = 2x \end{array} \right/ =$$

$$= 2\pi \int_0^1 x \sqrt{1 + 4x^2}\, dx = \left/ \begin{array}{l} t = 1 + 4x^2 \\ dt = 8x\, dx \\ \text{byter gränser} \end{array} \right/ =$$

$$= 2\pi \int_1^5 \sqrt{t}\, \frac{dt}{8} = \frac{\pi}{4} \left[\frac{t^{3/2}}{\frac{3}{2}} \right]_1^5 =$$

$$= \frac{\pi}{6} \left(5^{3/2} - 1 \right) = \frac{\pi}{6} \left(5\sqrt{5} - 1 \right) \text{ a.e.}$$

Ex. <u>Sfärens area.</u>



y = radien.

$$A = \int (\text{TP:s väg})\, ds = 2\int_0^R 2\pi y\, ds =$$

$$= 4\pi \int_0^R \sqrt{R^2-x^2}\sqrt{dx^2+dy^2} =$$

$$= 4\pi \int_0^R \sqrt{R^2-x^2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx =$$

$$y = \sqrt{R^2-x^2}$$

$$y' = \frac{1}{2}(R^2-x^2)^{-\frac{1}{2}}\cdot(-2x)$$

$$= 4\pi \int_0^R \sqrt{R^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}\, dx =$$

$$= 4\pi \int_0^R \sqrt{R^2-x^2}\sqrt{\frac{R^2\cancel{-x^2}\cancel{+x^2}}{R^2-x^2}}\, dx = \underline{\underline{4\pi R^2}}$$

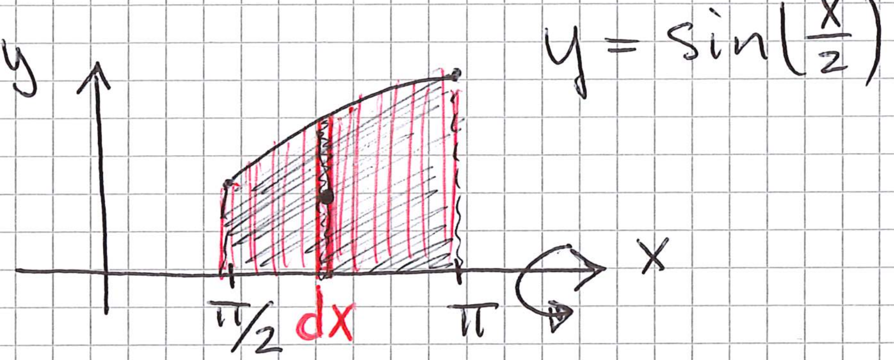
Blandade exempel.

Ex. Beräkna volymen av den kropp som bildas då ytan mellan x-axeln och

$y = \sin\left(\frac{x}{2}\right)$, $\pi/2 \le x \le \pi$

roterar ett varv kring x-axeln.

L.



$$V = \int (\text{TP:s väg})\ \text{arean} = \int 2\pi \frac{y}{2}\, y\, dx =$$

$$= \int \pi y^2\, dx = \int_{\pi/2}^{\pi} \pi \sin^2\left(\frac{x}{2}\right) dx =$$

Skivformeln.

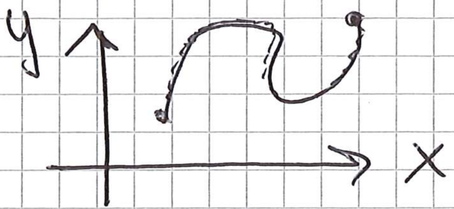
$$= \pi \int_{\pi/2}^{\pi} \frac{1-\cos x}{2}\, dx = \frac{\pi}{2}\Big[x - \sin x\Big]_{\pi/2}^{\pi} =$$

$$= \frac{\pi}{2}\left(\pi - \frac{\pi}{2} + 1\right) = \underline{\underline{\frac{\pi}{2}\left(\frac{\pi}{2}+1\right) \text{v.e.}}}$$

Ex. Beräkna längden av

$$\begin{cases} x(t) = \frac{3}{2}\sqrt{t} \\ y(t) = (t+1)^{3/2} \end{cases}, \quad 1 \leq t \leq 2.$$

L.



$$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$$

$$= \left/ \begin{array}{l} \frac{dx}{dt} = x'(t) = \frac{3}{4} t^{-1/2} = \frac{3}{4\sqrt{t}} \\ \frac{dy}{dt} = y'(t) = \frac{3}{2}(t+1)^{1/2} = \frac{3}{2}\sqrt{t+1} \end{array} \right/ =$$

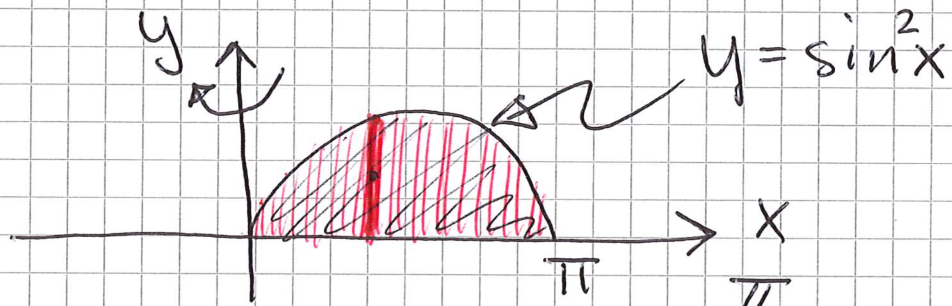
$$= \int_1^2 \sqrt{\frac{9}{16t} + \frac{9(t+1)}{4}}\, dt = \frac{3}{2}\int_1^2 \sqrt{\frac{1}{4t} + (t+1)}\, dt$$

$$= \frac{3}{2}\int_1^2 \sqrt{\frac{1+4t^2+4t}{4t}}\, dt = \frac{3}{2}\int_1^2 \sqrt{\frac{(2t+1)^2}{4t}}\, dt =$$

$$= \frac{3}{2}\int_1^2 \frac{2t+1}{2\sqrt{t}}\, dt = \frac{3}{2}\left[\frac{t^{3/2}}{\frac{3}{2}} + \sqrt{t}\right]_1^2 = \underline{\underline{\frac{7\sqrt{2}-5}{2}}} \text{ l.e}$$

Ex. Räkna ut volymen av den rotationskropp som bildas då ytan mellan kurvan $y = \sin^2 x$, $0 \le x \le \pi$, och x-axeln roteras ett varv kring y-axeln.

L.



$$V = \int \underbrace{2\pi x}_{TP\text{:s väg}} \underbrace{y\,dx}_{=\text{arean}} = 2\pi \int_0^{\pi} x \sin^2 x \, dx =$$

rörformeln

$$= 2\pi \int_0^{\pi} x \left(\frac{1 - \cos 2x}{2} \right) dx =$$

$$= \pi \int_0^{\pi} (x - x\cos 2x)\, dx \underset{p.i}{=} \pi \left[\frac{x^2}{2} \right]_0^{\pi} - \pi$$

$$\left(\underbrace{\left[\frac{\sin 2x}{2} x \right]_0^{\pi}}_{=0} - \int_0^{\pi} \frac{\sin 2x}{2} \cdot 1 \, dx \right) = \frac{\pi^3}{2} + \frac{\pi}{2} \left[\frac{-\cos 2x}{2} \right]_0^{\pi} =$$

$$= \frac{\pi^3}{2} \text{ v.e.}$$