Fö 6 Generaliserade integraler.

Läs 6.7.

Obegränsat intervall. $\int_a^{\infty} f(x)\,dx$

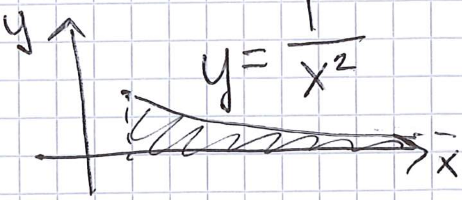
Def: Om $\lim_{\omega\to\infty} \int_a^{\omega} f(x)\,dx$ existerar

säg lika med A. så är

$\int_a^{\infty} f(x)\,dx$ konvergent med värdet A.

Annars divergent.

Ex. $\int_1^{\infty} \frac{1}{x^2}\,dx$



$$\int_1^{\omega} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\omega} = -\frac{1}{\omega}+1 \longrightarrow 1$$

då $\omega \to \infty$.

$\int_1^{\infty} \frac{1}{x^2}\,dx$ konvergent med värdet 1.

Ex. $\int_1^\infty \frac{1}{x}\,dx$ divergent.

$$\int_1^\omega \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^\omega = \ln\omega - \underbrace{\ln 1}_{=0} \to \infty$$

då $\omega \to \infty$.

$$\boxed{\text{Sats: } \int_1^\infty \frac{1}{x^\alpha}\,dx \ \text{konv.} \iff \alpha > 1}$$

Ex $\int_0^\infty \sin x\,dx$ divergent.

Ex. Beräkna (om konvergens).

$$\int_4^\infty \frac{1}{x^2-2x-3}\,dx$$

L: $$\int_4^\omega \frac{1}{x^2-2x-3}\,dx = \int_4^\omega \frac{1}{(x+1)(x-3)}\,dx =$$

PBU.

$$= \int_4^{\omega} \left(\frac{A}{x+1} + \frac{B}{x-3} \right) dx = \Bigg/ \text{ger } -\frac{1}{4} = A, \ \frac{1}{4} = B \Bigg/ =$$

$$= \frac{1}{4} \int_4^{\omega} \left(\frac{-1}{x+1} + \frac{1}{x-3} \right) dx =$$

$$= \frac{1}{4} \Big[\left(\ln|x-3| - \ln|x+1| \right) \Big]_4^{\omega} = \frac{1}{4} \left[\ln \left| \frac{x-3}{x+1} \right| \right]_4^{\omega} =$$

$$= \frac{1}{4} \left(\ln \frac{\omega-3}{\omega+1} - \ln \frac{1}{5} \right) = \Big/ \text{Dominerande faktor} \Big/$$

$$= \frac{1}{4} \left(\ln \frac{\cancel{\omega}(1 - \overset{\to 0}{\frac{3}{\omega}})}{\cancel{\omega}(1 + \underset{\to 0}{\frac{1}{\omega}})} - \ln 5^{-1} \right) \to \frac{\ln 5}{4}$$

då $\omega \to \infty$.

$\to \ln 1 = 0$

Svar: $\displaystyle\int_4^{\infty} \frac{1}{x^2-2x-3} \, dx = \frac{\ln 5}{4}$.

Ex. $\displaystyle\int_0^\infty x e^{-x}\,dx$ konv. med värdet 1.

L: $$\int_0^\omega x e^{-x}\,dx \underset{p.i}{=} \left[\frac{e^{-x}}{-1}\, x\right]_0^\omega - \int_0^\omega \frac{e^{-x}}{-1}\cdot 1\,dx =$$

$$= -\frac{\omega}{e^\omega} - 0 + \left[\frac{e^{-x}}{-1}\right]_0^\omega = \overset{\to 0}{-\frac{\omega}{e^\omega}} - \overset{\to 0}{\frac{1}{e^\omega}} + 1 \to 1$$

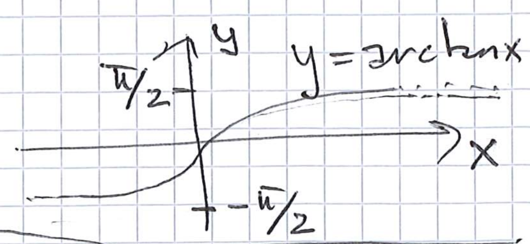
då $\omega \to \infty$.

Ex. $\displaystyle\int_0^\infty \frac{4}{4+x^2}\,dx = \pi$.

L. $$\int_0^\omega \frac{4}{4+x^2}\,dx = \int_0^\omega \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx =$$

$$= \left[\frac{\arctan\left(\frac{x}{2}\right)}{\frac{1}{2}}\right]_0^\omega = 2\left(\overset{\to \pi/2}{\arctan\left(\frac{\omega}{2}\right)} - 0\right)$$

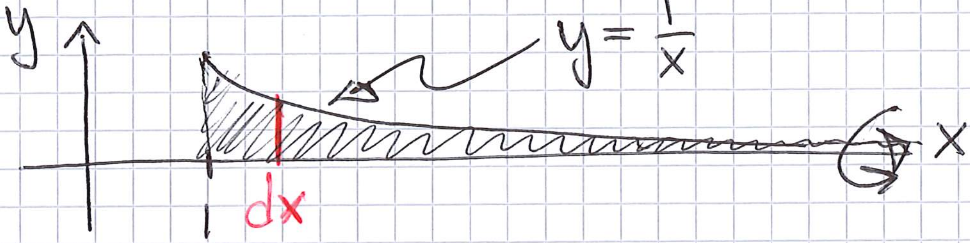
$\longrightarrow \pi$ då $\omega \to \infty$.



Rotationskropp.

Ex. $D = \{ (x,y) : 0 \leq y \leq \frac{1}{x} , x \geq 1 \}$

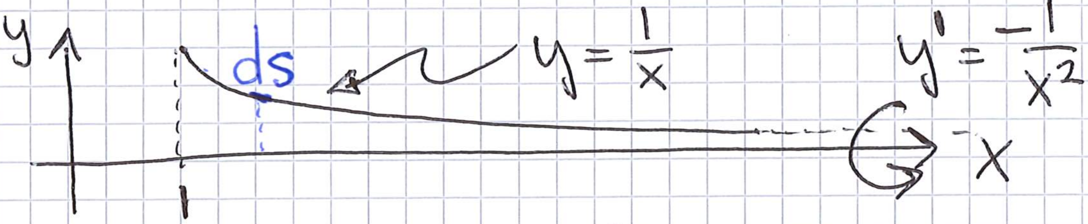
roteras ett varv kring x-axeln.



$$V = \int_1^\infty \pi y^2 dx = \pi \int_1^\infty \frac{1}{x^2} dx = \pi \left[-\frac{1}{x} \right]_1^\infty = \pi$$

Sök strutens mantelarea.

Paradox ??



$$A = \int_1^\infty 2\pi y \, ds = 2\pi \int_1^\infty y \sqrt{dx^2 + dy^2} =$$

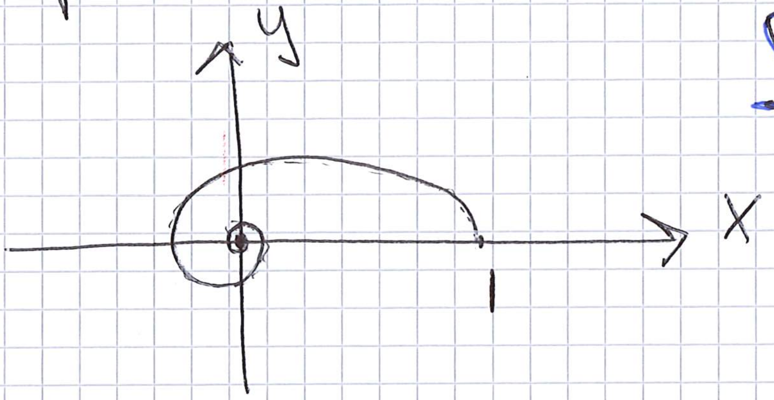
$$= 2\pi \int_1^\infty y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_1^\infty \frac{1}{x} \sqrt{\underbrace{1 + \frac{1}{x^4}}_{>1}} \, dx >$$

$$> 2\pi \int_1^\infty \frac{1}{x} dx = \infty$$

Ex. **Kurvlängd**

$$r = e^{-\varphi}, \quad \varphi \geq 0.$$

polära koordinater



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

obs! gäller alltid! vid p.k.

$$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int_0^\infty \sqrt{\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2}\, d\varphi$$

$$= \left/ \begin{array}{l} x = e^{-\varphi}\cos\varphi \\ y = e^{-\varphi}\sin\varphi \\ \frac{dx}{d\varphi} = -e^{-\varphi}\cos\varphi - e^{-\varphi}\sin\varphi \\ \quad ; \text{osv} \end{array} \right/ = \ldots =$$

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$$= \int_0^\infty \sqrt{2e^{-2\varphi}}\, d\varphi = \sqrt{2}\left[\frac{e^{-\varphi}}{-1}\right]_0^\infty = \sqrt{2} \text{ l.e.}$$

Obegränsad integrand.

Ex. $\displaystyle\int_0^1 \frac{1}{x^{2/3}}\,dx$

L: $\displaystyle\int_\varepsilon^1 \frac{1}{x^{2/3}}\,dx = \left[\frac{x^{-\frac{2}{3}+1}}{\frac{1}{3}}\right]_\varepsilon^1 =$

$\displaystyle = 3\left[x^{\frac{1}{3}}\right]_\varepsilon^1 = 3(1-\underbrace{\sqrt[3]{\varepsilon}}_{\to 0}) \to 3$

då $\varepsilon \to 0^+$

$\displaystyle\int_0^1 \frac{1}{x^{2/3}}\,dx$ konvergent med värdet 3.

Sats: $\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx$ konv. $\Leftrightarrow \alpha < 1$

Ex. Beräkna $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

L: $\int_\varepsilon^1 \frac{1}{\sqrt{x}}\cdot \ln x \, dx = \int_\varepsilon^1 \underset{\uparrow}{x^{-\frac{1}{2}}}\cdot \underset{\downarrow}{\ln x}\, dx \underset{p.i}{=}$

$= \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\ln x\right]_\varepsilon^1 - \int_\varepsilon^1 2\sqrt{x}\cdot\frac{1}{x}\,dx =$

$= -2\sqrt{\varepsilon}\ln\varepsilon - 2\left[\frac{x^{1/2}}{\frac{1}{2}}\right]_\varepsilon^1 =$

*

$= -2\overbrace{\sqrt{\varepsilon}\ln\varepsilon}^{\to 0} - 4(1-\overbrace{\sqrt{\varepsilon}}^{\to 0}) \to -4$

då $\varepsilon \to 0^+$

* Se här:

$\lim_{\varepsilon\to 0^+} \sqrt{\varepsilon}\ln\varepsilon = \left/ \begin{matrix} t=\ln\varepsilon \Leftrightarrow \varepsilon = e^t \\ \varepsilon\to 0^+ \Rightarrow t\to -\infty \end{matrix} \right/ =$

$= \lim_{t\to-\infty} e^{\frac{t}{2}} t = \lim_{t\to-\infty} \frac{t}{e^{-t/2}} \underset{\substack{\uparrow \\ \text{hastighetstabell}}}{=} 0$

Svar: $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = -4$

Ex. Beräkna $\displaystyle\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx$

$$L: \int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx = \overset{\text{II.}}{\int_0^1 \frac{dx}{(x+1)\sqrt{x}}} + \overset{\text{I.}}{\int_1^\infty \frac{dx}{(x+1)\sqrt{x}}}$$

I. $$\int_1^\omega \frac{1}{(x+1)\sqrt{x}}\,dx = \left/ \begin{array}{l} t=\sqrt{x} \overset{(x>0)}{\iff} x=t^2 \\ \frac{dx}{dt}=2t \end{array} \right/ =$$

$$= \int_1^{\sqrt{\omega}} \frac{1}{(t^2+1)\,t}\,2t\,dt = \int_1^{\sqrt{\omega}} \frac{2}{t^2+1}\,dt =$$

$$= 2\left[\arctan t\right]_1^{\sqrt{\omega}} \to 2\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

då $\omega \to \infty$

II. $$\ldots = 2\left[\arctan t\right]_{\sqrt{\varepsilon}}^1 \to 2\left(\frac{\pi}{4}-0\right) = \frac{\pi}{2}$$

då $\varepsilon \to 0^+$

Svar: Konv. värdet $= \pi$.

Något om serier.

Ex. $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$

Ex. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$

Integraler / summor. släkt.

Viktigt.

$\boxed{\text{Geometrisk summa}}$ (serie)

konstant kvot mellan konsekutiva termer.

$a + ak + ak^2 + \ldots + ak^{n-1} = a\frac{1-k^n}{1-k}$

låt $n \to \infty$. Se boken. Sid 41

$|k| < 1$ ger konvergens