

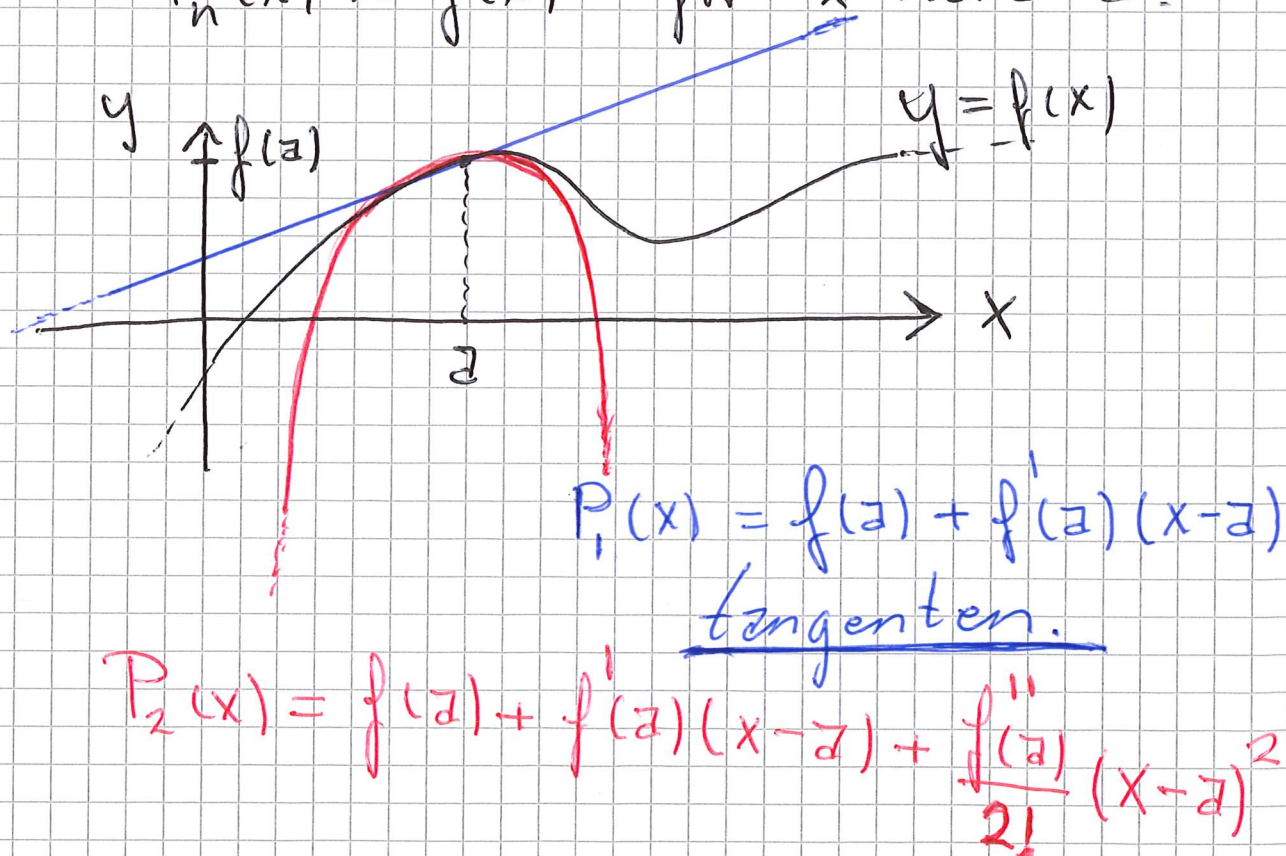
Fö7 | Maclaurin- och Taylorutveckling

Läs 8.1-8.4.

Approximera $y=f(x)$ i närheten av $(a, f(a))$ med ett s.k. taylorpolynom, $P_n(x)$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$P_n(x) \approx f(x)$ för x nära a .



Om $z=0$ kallas $P_n(x)$ maclaurinpolynom.

n = ordningen.

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$n!$ utläses n fakultet.

$$n! = n(n-1)(n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

t.ex. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$P_n(x)$ approximerar $f(x)$ för x nära
Maclaurin noll.

$$\text{felet} = |f(x) - P_n(x)| = B(x) \cdot x^{n+1}$$

↑
Se senare.

där $B(x)$ är begränsad i
någon omgivning av 0.

skrivs $O(x^{n+1})$, n heltal ≥ 0

utläses start order av x^{n+1}

Ex. $f(x) = \sin 3x$

mac|aurinutv.

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) = \sin 3x, \quad f(0) = 0$$

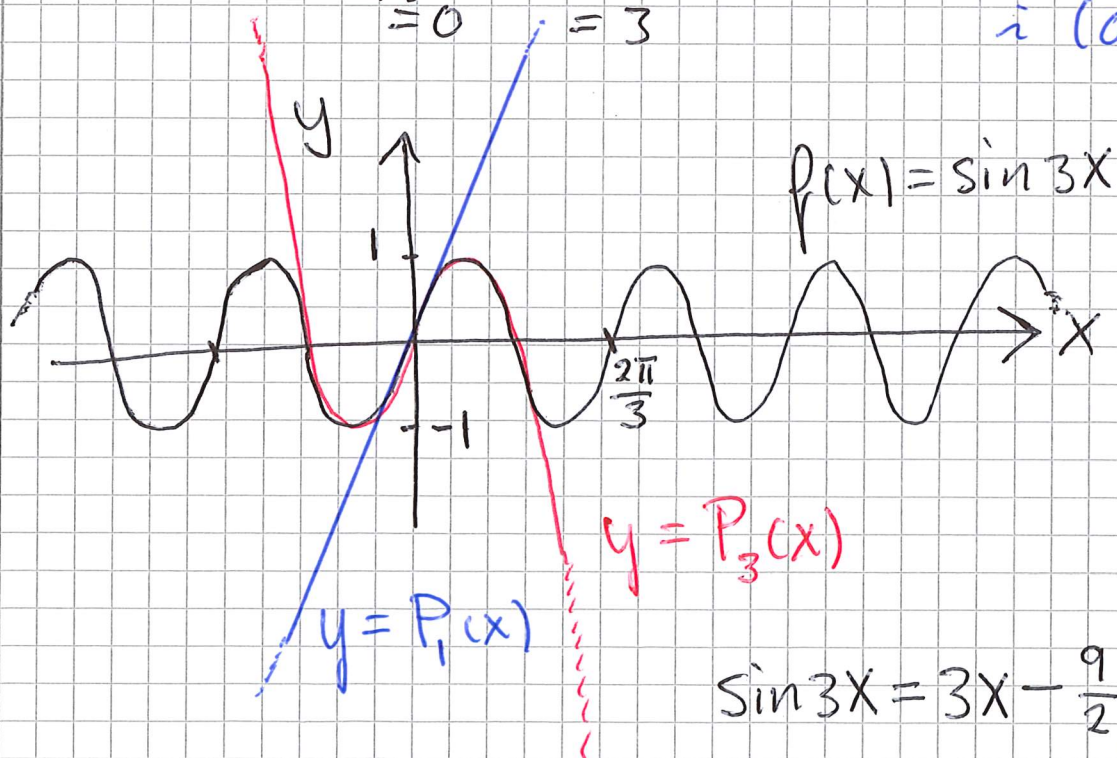
$$f'(x) = 3 \cos 3x, \quad f'(0) = 3$$

$$f''(x) = -9 \sin 3x, \quad f''(0) = 0$$

$$f^{(3)}(x) = -27 \cos 3x, \quad f^{(3)}(0) = -27$$

$$P_3(x) = 3x - \frac{27}{6}x^3 = 3x - \frac{9}{2}x^3$$

$$P_1(x) = \underbrace{f(0)}_{=0} + \underbrace{f'(0)}_{=3}x = 3x \quad \text{tangenten in } (0,0).$$



$$\sin 3x = 3x - \frac{9}{2}x^3 + O(x^5)$$

Ex. Beräkna $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$ typ $\left[\frac{0}{0}\right]$

L: $f(x) = e^x$ $f'(x) = f''(x) = \dots = e^x$

$$e^x = \underbrace{f(0)}_{=1} + \underbrace{f'(0)}_{=1}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + O(x^{n+1})$$

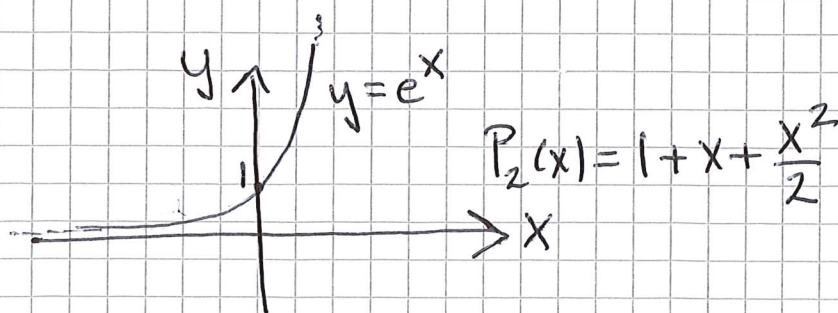
$$e^x = 1 + x + \frac{x^2}{2} + O(x^3) \text{ för } x \text{ nära } 0.$$

$$\begin{aligned} \frac{e^x - 1 - x}{x^2} &= \frac{1 + x + \frac{x^2}{2} + O(x^3) - 1 - x}{x^2} = \\ &= \frac{\frac{x^2}{2} + O(x^3)}{x^2} = \frac{x^2 \left(\frac{1}{2} + O(x) \right)}{x^2} = \frac{1}{2} + O(x) \xrightarrow{x \rightarrow 0} \frac{1}{2} \end{aligned}$$

$O(x) \rightarrow 0$ då $x \rightarrow 0$.
 $O(x) = B(x) \cdot x$
 $B(x)$ begränsad.

Svar: $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$

typ $\left[\frac{0}{0}\right]$



Ex. Bestäm Maclaurinutvecklingen
av ordning 5 med restterm
i ordoform av $f(x) = \cos x$

L: Maclaurinpolynom $P_n(x)$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) = \cos x, \quad f'(x) = -\sin x$$

$$f''(x) = -\cos x, \quad f^{(3)}(x) = \sin x$$

$$f^{(4)}(x) = \cos x \quad \dots \text{ osv.}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$$

$$\text{Svar: } \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24} \approx \cos x \quad \text{för } x \text{ nära } 0.$$

Obs!

Elementära Maclaurinutr. (sid 360)

lär vi oss utantill. (eller?)

$$f(x) = P_n(x) + \mathcal{O}(x^{n+1})$$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Se sid 360

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \mathcal{O}(x^9)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \mathcal{O}(x^8)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \text{ se boken.}$$

Ex. Bestäm Maclaurinutvecklingen
av ordning 2 med restterm
i ordform av
 $f(x) = e^{2x} / \ln(1+x)$

$$L: f(x) = e^{2x} / \ln(1+x) = \left/ \begin{array}{l} e^t = 1 + t + \frac{t^2}{2!} + \dots \\ t = 2x \\ \ln(1+x) = \dots \end{array} \right/ \quad \text{Sid 360}$$

$$= \left(1 + 2x + \frac{(2x)^2}{2!} + O(x^3) \right) \left(x - \frac{x^2}{2} + O(x^3) \right) =$$

$$= x - \frac{x^2}{2} + 2x^2 + O(x^3) = x + \frac{3}{2}x^2 + O(x^3)$$

$O(x^3)$ slukar in alla termer
av grad 3 och högre.

$$\text{t.ex. } O(x^3) = x^3 + O(x^4) + 2x^3 - x^4 + \dots =$$

$$= O(x^3)$$

↑ "ny" start order av x^3

obs! Maclaurin dvs x nära noll.

Ex. $\lim_{x \rightarrow 0} \frac{1 - \cos x^2}{x \sin^3 x}$

L: Sid 360: $\left/ \begin{array}{l} \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \\ t = x^2 \end{array} \right/$

$$\begin{aligned} \frac{1 - \cos x^2}{x \sin^3 x} &= \frac{1 - \left(1 - \frac{(x^2)^2}{2} + O(x^8)\right)}{x (x + O(x^3))^3} \\ &= \frac{\frac{x^4}{2} + O(x^8)}{x^4 + O(x^6)} = \frac{\frac{1}{2} + \overset{\rightarrow 0}{O(x^4)}}{1 + \overset{\rightarrow 0}{O(x^2)}} \rightarrow \frac{1}{2} \end{aligned}$$

$\xrightarrow{0} \frac{d^0}{dx^0} x \rightarrow 0$

Svar: $\lim_{x \rightarrow 0} \frac{1 - \cos x^2}{x \sin^3 x} = \frac{1}{2}$

Ex. Beräkna maclaurinpolynomet av grad 3 till

$$f(x) = \frac{1}{\sqrt{1-2x}}$$

$$L: \frac{1}{\sqrt{1-2x}} = (1-2x)^{-\frac{1}{2}} = (1+(-2x))^{-\frac{1}{2}} =$$

$$= \frac{(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3}{\alpha = -\frac{1}{2}}$$

Sätt in $-2x$ på x plats

$$= 1 - \frac{1}{2}(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-2x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3 \cdot 2 \cdot 1}(-2x)^3 + O(x^4)$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + O(x^4)$$

$$\text{Svar: } P_3(x) = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3$$

Ex. Beräkna gränsvärdet

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

$$L: \frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} =$$

$$= \frac{x - \left(x - \frac{x^3}{3!} + O(x^5) \right)}{x(x + O(x^3))} =$$

$$= \frac{\frac{x^3}{6} + O(x^5)}{x^2 + O(x^4)} = \frac{\cancel{x^2} \left(\frac{x}{6} + O(x^3) \right)}{\cancel{x^2} (1 + O(x^2))} =$$

$$= \frac{\overset{\rightarrow 0}{\frac{x}{6}} + \overset{\rightarrow 0}{O(x^3)}}{1 + \overset{\rightarrow 0}{O(x^2)}} \rightarrow 0 \quad \text{da } x \rightarrow 0$$

$$\text{Satz: } \lim_{x \rightarrow 0} = 0$$
