Fö 8 Läs 8.4.

Maclaurins formel.

Antag. f är $n+1$ ggr kont. deriverbar i någon omgivning av 0.

$$f(x) = f(0) + \int_0^x f'(t)\,dt = f(0) + \int_0^x 1 \cdot f'(t)\,dt \underset{p.i}{=}$$

$$\underset{p.i}{=} f(0) + \Big[(t-x) f'(t)\Big]_0^x - \int_0^x (t-x) f''(t)\,dt \underset{p.i}{=}$$

$$= f(0) + f'(0)x - \left[\frac{(t-x)^2}{2} f''(t)\right]_0^x + \int_0^x \frac{(t-x)^2}{2} f^{(3)}(t)\,dt =$$

$$= \ldots\ldots = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots$$

$$\ldots + \frac{f^{(n)}(0)}{n!}x^n + \int_0^x \frac{(t-x)^n}{n!} f^{(n+1)}(t)\,dt$$

(n jämn)

$$f(x) = P_n(x) + \underbrace{\int_0^x \frac{(t-x)^n}{n!} f^{(n+1)}(t)\,dt}_{\text{Resttermen}}$$

$$\underbrace{\text{Resttermen}}_{=R_{n+1}(x)} = \int_0^x \frac{(t-x)^n}{n!} f^{(n+1)}(t)\,dt \underset{\text{i.m.s.}}{=}$$

$$= f^{(n+1)}(\xi)\int_0^x \frac{(t-x)^n}{n!}\,dt = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

$0 \le \xi \le x$.

Lagranges form.

$$R_{n+1}(x) = B(x)\,x^{n+1} = \mathcal{O}(x^{n+1})$$

B = begränsad.

Ex. Maclaurinutv. ordning 2.

Alt 1

$$f(x) = e^x \sin x$$

$$f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

$$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$$

$$e^x \sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \mathcal{O}(x^3)$$

$$= x + x^2 + \mathcal{O}(x^3)$$

Alt 2. $e^x \sin x = \Big/ \begin{matrix} \text{Standardutv.} \\ \text{sid 360} \end{matrix} \Big/ =$

$$= \left(1 + x + \frac{x^2}{2!} + \mathcal{O}(x^3)\right)\left(x + \mathcal{O}(x^3)\right) =$$

$$= x + x^2 + \mathcal{O}(x^3)$$

Räkning med stort ordo.

$$\mathcal{O}(x^2) + 5x^2 + \mathcal{O}(x^4) - x^3 = \mathcal{O}(x^2)$$

"ny" stort ordo.
slukar alla termer av
grad 2 och högre.

$$\mathcal{O}(x^2) - \mathcal{O}(x^2) = \mathcal{O}(x^2)$$

$$-\mathcal{O}(x^n) = \mathcal{O}(x^n)$$

$$\mathcal{O}(x^n)\mathcal{O}(x^m) = \mathcal{O}(x^{n+m})$$

för x nära noll.

obs!

Ex. $\lim_{x\to 1} \frac{\ln x^2 - (\ln x)^2}{x - \sqrt{x}} = \Big/$ Sub. $t = x-1$, $x = 1+t$, $x \to 1 \Rightarrow t \to 0 \Big/$

typ $\left[\frac{0}{0}\right]$

$$= \lim_{t\to 0} \frac{2\ln(1+t) - (\ln(1+t))^2}{1+t-(1+t)^{1/2}} =$$

$\uparrow$ Sid 360.

$$= \lim_{t\to 0} \frac{2(t+\mathcal{O}(t^2)) - (t+\mathcal{O}(t^2))^2}{1+t-(1+\frac{1}{2}t+\mathcal{O}(t^2))} =$$

$$= \lim_{t\to 0} \frac{2t+\mathcal{O}(t^2)}{\frac{1}{2}t+\mathcal{O}(t^2)} =$$

$$= \lim_{t\to 0} \frac{\cancel{t}(2+\overbrace{\mathcal{O}(t)}^{\to 0})}{\cancel{t}(\frac{1}{2}+\underbrace{\mathcal{O}(t)}_{\to 0})} = 4$$

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t.ex. $x \to 4$ Sub. $t = x-4$

Om $x \not\to 0$ Gör Sub.

$x \to \pi$ Sätt $t = x - \pi$

$x \to \infty$ Sätt $t = \frac{1}{x}$

Ex. Beräkna

$$\lim_{x\to\infty} \left(x\sin\frac{1}{x} - 1\right)x^2$$

typ $[0\cdot\infty]$

L. $\left/ \text{Sub. } t = \frac{1}{x} \Leftrightarrow x = \frac{1}{t} \quad x\to\infty \Rightarrow t\to 0 \right/$

$$\left(x\sin\frac{1}{x} - 1\right)x^2 = \left(\frac{1}{t}\sin t - 1\right)\frac{1}{t^2} =$$

$$= \left(\frac{1}{t}\left(t - \frac{t^3}{3!} + \mathcal{O}(t^5)\right) - 1\right)\frac{1}{t^2} =$$

$$= \left(1 - \frac{t^2}{6} + \mathcal{O}(t^4) - 1\right)\frac{1}{t^2} =$$

$$= -\frac{1}{6} + \underbrace{\mathcal{O}(t^2)}_{\to 0} \longrightarrow -\frac{1}{6}$$

då $t\to 0$.

Svar: $\lim_{x\to\infty} \left(x\sin\frac{1}{x} - 1\right)x^2 = -\frac{1}{6}$

Entydighetssatsen. Se boken sid 368.

Ex. Bestäm Maclaurinutvecklingen av ordning 4.

$\to 0$ då $x \to 0$ obs!

$$e^{\cos x} = e^{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)\right)} =$$

$$= e \cdot e^{\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)\right)} =$$

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

$$t = -\frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

obs! $t \to 0$ då $x \to 0$

$$= e\left(1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \frac{\left(-\frac{x^2}{2} + O(x^4)\right)^2}{2} + O(x^6)\right)$$

$$= e\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^4}{8} + O(x^6)\right) =$$

$$= e - \frac{e}{2}x^2 + \frac{e}{6}x^4 + O(x^6)$$

Ex. Undersök om funktionen

$$f(x) = \ln(\cos x) + \frac{1}{2}\sin x^2$$

har lokalt maximum eller minimum i $x=0$.

L. Maclaurinutv.

läs här $\ln(1+ \ldots)$

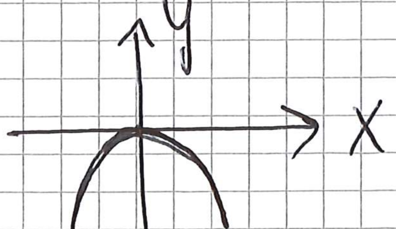
obs!

$$f(x) = \ln\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)\right) + \frac{1}{2}\sin x^2 =$$

$$= \left| \begin{array}{l} \ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ldots \\ t = -\frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \\ \sin t = t - \frac{t^3}{3!} + \ldots, \quad t = x^2 \end{array} \right| =$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{\frac{x^4}{4}}{2} + O(x^6) + \frac{1}{2}x^2 + O(x^6) =$$

$$= -\frac{1}{12}x^4 + O(x^6) \quad \text{för } x \text{ nära } 0.$$



Svar: f har lokalt maximum i $x=0$.

Ex. Bestäm alla värden på konstanten a för vilka gr.v

$$\lim_{x\to 0} \frac{\ln(1+ax) - 2x + \arctan(ax)}{x^2}$$

existerar, samt ange i dessa fall gränsvärdet.

L: $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots$

$t = ax$.

$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \dots$

$$= \frac{ax - \frac{(ax)^2}{2} + \mathcal{O}(x^3) - 2x + ax + \mathcal{O}(x^3)}{x^2} =$$

$$= \frac{(2a-2)x - \frac{a^2x^2}{2} + \mathcal{O}(x^3)}{x^2} =$$

$$= \frac{2(a-1)}{x} - \frac{a^2}{2} + \mathcal{O}(x) \overset{a=1}{=} -\frac{1}{2} + \underbrace{\mathcal{O}(x)}_{\to 0} \to -\frac{1}{2}$$

då $x \to 0$

Gr.v exist. $\iff$ $a = 1$

Om $a \neq 1$

$\frac{2(a-1)}{x} \to \pm\infty$

då $x \to 0$

Ex. Räkna ut $f^{(10)}(0)$ och $f^{(11)}(0)$

då $f(x) = x \sin x^2$

L. $f(x) = P_n(x) + \mathcal{O}(x^{n+1}) =$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(10)}(0)}{10!}x^{10} + \frac{f^{(11)}(0)}{11!}x^{11} + \dots$$

$$\dots \frac{f^{(n)}(0)}{n!}x^n + \mathcal{O}(x^{n+1}) =$$

$$= x\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots\right)$$

$$= x^3 - \frac{1}{6}x^7 + \frac{1}{5!}x^{11} - \frac{1}{7!}x^{15} + \dots$$

identifiering ger

koeff. x^{10} : $f^{(10)}(0) = 0$

x^{11} : $\frac{f^{(11)}(0)}{11!} = \frac{1}{5!}$

Svar: $f^{(10)}(0) = 0$, $f^{(11)}(0) = \frac{11!}{5!}$

Maclaurinserier.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Derivera.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Geometrisk serie. , $|x| < 1$.

* $$1 - x + x^2 - x^3 + x^4 - \ldots = \frac{1}{1+x}$$

Integrera.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Sub. i *

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

Integrera.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Obs! Konvergensområde ??