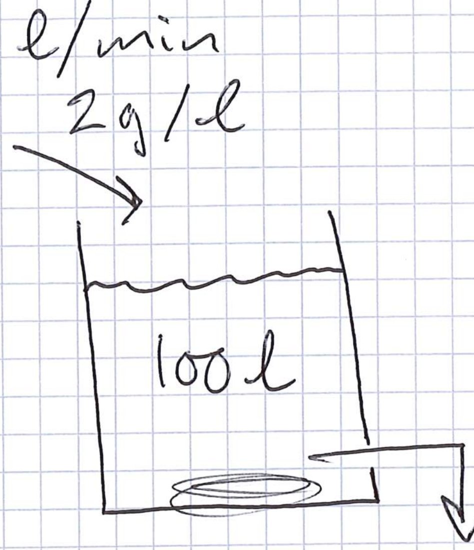
Fö 9 | Läs 9.1–9.2.

Differentialekvationer.

Ex.



$y(t)$ = mängden salt vid tidpunkt t.

$$y(0) = 0$$

$$y'(t) = \underbrace{3 \cdot 2}_{\text{in}} - \underbrace{3 \cdot \frac{y(t)}{100}}_{\text{ut}}$$

↑ förändringen

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i.f-metoden

linjär diff.ekv av 1:a ordningen

Lös $y'(x) + g(x)\,y(x) = h(x)$.

g, h givna.

L: $\text{i.f} = e^{G(x)}$

/ i.f utläses integrerande faktor

$$y'(x) + g(x)\,y(x) = h(x)$$

obs! y' "ensam".

$$\Big/ \text{ mult. med } i \cdot f = e^{G(x)} \Big/$$

$$\Leftrightarrow \underbrace{y'(x)e^{G(x)} + g(x)e^{G(x)}y(x)}_{\text{= derivatan av en produkt.}} = h(x)e^{G(x)}$$

Alltid produkten av sökta funktionen $y(x)$ och $i.f$.

$$\Leftrightarrow \left(y(x)e^{G(x)}\right)' = h(x)e^{G(x)}$$

Gör kontroll.

$$\Leftrightarrow \Big/ \int \left(y(x)e^{G(x)}\right)' = \int h(x)e^{G(x)} \Big/$$

$$y(x)e^{G(x)} = \int h(x)e^{G(x)}\,dx + C$$

Lös ut $y(x)$.

obs! Glöm inte $+C$

Ex. Lös diff. ekv.

$$xy' + x^2 y = x^2 \quad , \quad x > 0 .$$

L: i.f-metoden.

$$xy' + x^2 y = x^2$$

$$y' + xy = x$$

$$\Big/ \; i.f = e^{\int x\,dx} = e^{\frac{x^2}{2}} \; \Big/ \quad \text{mult. med i.f.}$$

$$\underbrace{y' e^{\frac{x^2}{2}} + x e^{\frac{x^2}{2}} y}_{\text{derivatan av en produkt}} = x e^{\frac{x^2}{2}}$$

$$\left(y e^{\frac{x^2}{2}}\right)' = x e^{\frac{x^2}{2}}$$

$$y e^{\frac{x^2}{2}} = \int x e^{\frac{x^2}{2}}\,dx = \left/ \begin{array}{l} t = \frac{x^2}{2} \\ dt = x\,dx \end{array} \right/ = \int e^t\,dt =$$

$$= e^t + C = e^{\frac{x^2}{2}} + C$$

Svar: $y = 1 + C e^{-\frac{x^2}{2}}$ obs! lösningsskara

jfr riktningsfält se boken

Ex. Lös $\begin{cases} y'(x) = x - y(x) \\ y(0) = 0 \end{cases}$

L: Alltid diff. ekv först.

$$y' + y = x$$

$$/ \; i.f = e^{\int 1\,dx} = e^{x} \; /$$

$$\Leftrightarrow \quad y'e^{x} + e^{x}y = xe^{x}$$

$$\Leftrightarrow \quad (ye^{x})' = xe^{x}$$

$$\Leftrightarrow \quad ye^{x} = \int xe^{x}\,dx \underset{p.i}{=} xe^{x} - e^{x} + C$$

$$\Leftrightarrow \quad y = x - 1 + Ce^{-x}$$ nu till extra villkoret.

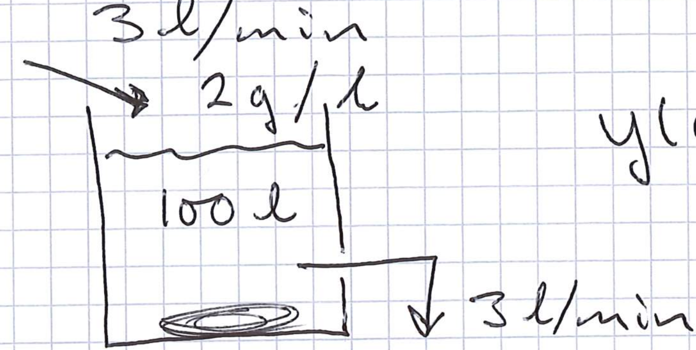
$y(0) = 0$ ger $0 = 0 - 1 + Ce^{0}$

$$C = 1$$

Svar: $y = x - 1 + e^{-x}$

Gör kontroll.

forts. inledande exempel.



$y(0) = 0$

$$y'(t) = 3 \cdot 2 - 3 \cdot \frac{y(t)}{100}$$

$$y'(t) + 0{,}03\,y(t) = 6$$

/ mult. med i.f $= e^{0,03t}$ /

$$y'(t)e^{0,03t} + 0{,}03e^{0,03t}y(t) = 6e^{0,03t}$$

$$\left(y(t)e^{0,03t}\right)' = 6e^{0,03t}$$

$$y(t)e^{0,03t} = \int 6e^{0,03t}\,dt$$

$$y(t)e^{0,03t} = 200e^{0,03t} + C$$

$y(0) = 0$ ger $C = -200$

$$y(t) = 200 - \frac{200}{e^{0,03t}}, \quad t \geq 0$$

Ex. Lös diff. ekv.

$$4y' - xy = 0$$

L: $y' - \frac{x}{4} y = 0$

Sök i.f. $= e^{\int -\frac{x}{4}\,dx} = e^{-\frac{x^2}{8}}$

mult med i.f. ger

$$y' e^{-\frac{x^2}{8}} - \frac{x}{4} e^{-\frac{x^2}{8}} y = 0$$

$$\left(y e^{-\frac{x^2}{8}}\right)' = 0 \qquad \text{integrera!}$$

$$y e^{-\frac{x^2}{8}} = C$$

$$y = C e^{\frac{x^2}{8}} \qquad \underline{\text{kontrollera!}}$$

Ex. Lös diff. ekv

$$y'(x) - \frac{y(x)}{x} = \frac{1}{x+1}, \quad x > 0$$

L. $y' - \frac{1}{x} y = \frac{1}{x+1}$ $\ast$, $x > 0$.

$i.f = e^{G(x)} = e^{-\ln x} = \frac{1}{x}$ $\Bigg/ \begin{array}{l} g(x) = -\frac{1}{x} \\ G(x) = -\ln|x| \end{array} \Bigg/$

obs! Förenkla.

läs $y' + \underbrace{\left(-\frac{1}{x}\right)}_{=g(x)} y = \underbrace{\frac{1}{x+1}}_{=h(x)}$.

mult. $\ast$ med i.f.

$$y' \frac{1}{x} - \frac{1}{x^2} y = \frac{1}{x(x+1)}$$

$$\left(y \cdot \frac{1}{x}\right)' = \frac{1}{x(x+1)}$$

Alltid produkten av sökta funktionen och i.f.

$$y \frac{1}{x} = \int \frac{1}{x(x+1)} dx \underset{\text{PBU}}{=} \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

Svar: $y = x\left(\ln\left|\frac{x}{x+1}\right| + C\right)$

Tillämpningar.

Ex. Radioaktivt sönderfall.

$m(t) =$ mängden (radon).

$m'(t) = -\lambda m(t) \quad , \quad \lambda > 0$

$m(0) = 3$ mg

$T_{1/2} = 3{,}8$ dygn. $\qquad m(7) = ?$

L: $m' + \lambda m = 0$

$/ \; i \cdot f = e^{\int \lambda dt} = e^{\lambda t} \; /$

$m' e^{\lambda t} + \lambda e^{\lambda t} m = 0$

$(m e^{\lambda t})' = 0$

$m e^{\lambda t} = \int 0 \, dt = C$

$m = \dfrac{C}{e^{\lambda t}} \iff m(t) = Ce^{-\lambda t}$

$m(0) = 3$ ger $3 = Ce^{0} \Rightarrow C = 3$.

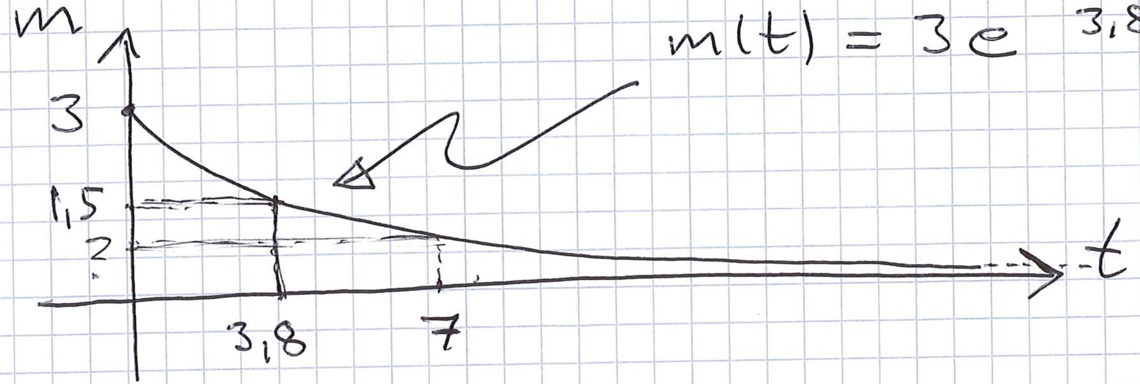
$m(3{,}8) = 1{,}5$ ger $1{,}5 = 3e^{-\lambda 3{,}8}$

$\iff \lambda = \dfrac{\ln(\frac{1}{2})}{-3{,}8} = \dfrac{\ln 2}{3{,}8} \quad \Big| \quad m(7) = \ldots$

Svar:

$$m(7) = 3e^{-\frac{\ln 2}{3,8}\cdot 7}$$

$$m(t) = 3e^{-\frac{\ln 2}{3,8}t}$$



Ex. Finn den lösning till diff. ekv $(1+x^2)y' + y = 2$ som har gränsvärdet 3 då $x \to \infty$.

L: $$(1+x^2)y' + y = 2$$

$$y' + \frac{1}{1+x^2}y = \frac{2}{1+x^2}$$

$$/ \; i.f = e^{\int \frac{1}{1+x^2}dx} = e^{\arctan x} \; /$$

$$y'e^{\arctan x} + \frac{1}{1+x^2}e^{\arctan x}y = \frac{2}{1+x^2}e^{\arctan x}$$

$$(y e^{\arctan x})' = \frac{2}{1+x^2}e^{\arctan x}$$

$$y e^{\arctan x} = \int \frac{2}{1+x^2} e^{\arctan x} dx =$$

$$\left/ \begin{array}{l} t = \arctan x \\ \frac{dt}{dx} = \frac{1}{1+x^2} \end{array} \right/ = \int 2 e^t dt = 2e^t + C$$

Glöm inte $+C$.

$$y e^{\arctan x} = 2 e^{\arctan x} + C$$

$$y = 2 + C e^{-\arctan x}$$

$$\lim_{x \to \infty} \left(2 + C e^{-\overset{\to \pi/2}{\arctan x}}\right) = 2 + C e^{-\frac{\pi}{2}} = 3$$

givet.

$$C = e^{\pi/2}$$

Svar: $y = 2 + e^{\frac{\pi}{2} - \arctan x}$

Kom ihåg! Kontroll.

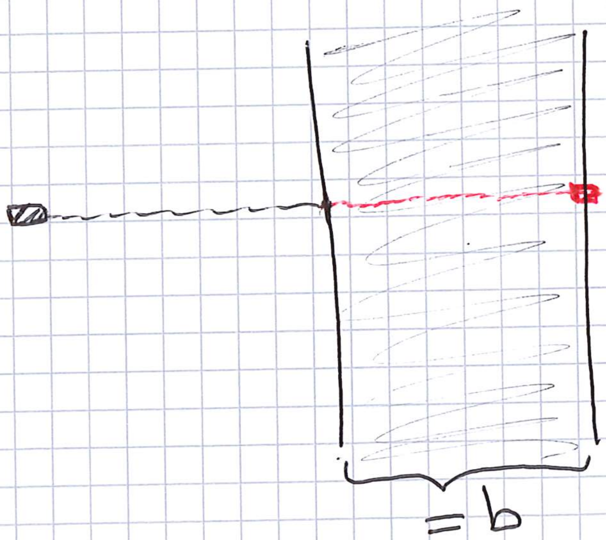
Ex. $v(t)$ = hastighet.

$$\frac{dv}{dt} = v'(t) = a(t) = \text{acceleration}.$$

$$s'(t) = v(t)$$

$$s(t) = \int v(t)\,dt \quad , \quad s(t) = \text{sträcka}.$$

Ex.



$v(0) = v_0$

retardationen är proportionell mot kulans hastighet.

$$v'(t) = -kv(t) \quad , \quad k>0$$

$$v(t) = v_0 e^{-kt}$$

$$s(t) = \int v_0 e^{-kt}\,dt = \frac{v_0 e^{-kt}}{-k} + C$$

$s \to b$ då $t \to \infty$. ger $C = b$

Ex. Lussebulle.

$T(t)$ = bullens temp.

$$\begin{cases} T'(t) = -k(T(t)-20) \\ T(0) = 200 \\ T(1) = 152 \end{cases}$$

När är $T = 35$?

L: $T' + kT = 20k$ $\quad / \text{i.f} = e^{kt} /$

$$T'e^{kt} + ke^{kt}T = 20ke^{kt}$$

$$(Te^{kt})' = 20ke^{kt}$$

$$Te^{kt} = \int 20ke^{kt}\,dt = 20e^{kt} + C$$

$T = 20 + Ce^{-kt}$, $T(0)=200$ ger $C = 180$

$T(1) = 152$ ger

$$152 = 20 + 180e^{-k} \Rightarrow k = \frac{\ln\left(\frac{132}{180}\right)}{-1}$$

$$35 = 20 + 180e^{t\ln\left(\frac{132}{180}\right)}$$

Svar: $t = \dfrac{\ln\left(\frac{15}{180}\right)}{\ln\left(\frac{132}{180}\right)}$

Ex. Lös $\begin{cases} y' - \frac{2}{x+1}y = (x+1)^3 \\ y(0) = 0 \end{cases}, \quad x > -1$

L. $y' - \frac{2}{x+1}y = (x+1)^3$

$$\text{i.f} = e^{\int -\frac{2}{x+1}dx} = e^{-2\ln(x+1)} = \frac{1}{(x+1)^2}$$

Obs! lätt att missa minustecknet.

mult. med i.f.

$$y'\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}y = x+1$$

$$\left(y\frac{1}{(x+1)^2}\right)' = x+1$$

$$y\frac{1}{(x+1)^2} = \int (x+1)\,dx = \frac{(x+1)^2}{2} + C$$

$$y = \frac{(x+1)^4}{2} + C(x+1)^2$$

$y(0) = 0$ ger $0 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$

Svar: $y = \frac{(x+1)^2}{2}(x^2+2x), \quad x > -1$

Ex. Lös diff. ekv.

$$(\cos x)\, y' = 2(\sin x)\, y + 2(\cos x + 1)$$

där $0 < x < \pi/2$

L. $$y' - \frac{2\sin x}{\cos x}\, y = 2\left(1 + \frac{1}{\cos x}\right)$$

$$\text{i.f} = e^{\int \frac{-2\sin x}{\cos x}dx} = e^{2\ln(\cos x)} = \cos^2 x$$

$$y'\cos^2 x - 2\sin x\cos x\, y = 2(\cos^2 x + \cos x)$$

$$(y\cos^2 x)' = 2(\cos^2 x + \cos x)$$

$$y\cos^2 x = \int 2\left(\frac{1+\cos 2x}{2} + \cos x\right)dx$$

$$y\cos^2 x = x + \frac{\sin 2x}{2} + 2\sin x + C$$

$$y = \frac{x}{\cos^2 x} + \tan x + \frac{2\sin x}{\cos^2 x} + \frac{C}{\cos^2 x}$$

hel lösningsskara, $0 < x < \pi/2$