LINKÖPINGS
UNIVERSITET

LÖSNINGAR TILL TENTAMEN I TAMS22 SANNOLIKHETSTEORI OCH BAYESIANSKA NÄTVERK, MÅNDAG 10 JANUARI 2022 KL 14.00-18.00.

1. (a) Define the binary valued r.v.s: $B =$ "battery charged/low", $D =$ "drops/does not drop ball", and $R =$ "reports/does not report a dropped ball". The following DAG describes the situation:

$$\begin{tikzcd} B \arrow[r] & D \arrow[r] & R \end{tikzcd}$$

The pmf for B is: $p_B(1) = 0.95$, $p_B(0) = 0.05$. The tables for the conditional pmfs are:

$$p_{D|B} = \begin{array}{c|cc} D\backslash B & 0 & 1 \\ \hline 0 & 0.1 & 0.99 \\ 1 & 0.9 & 0.01 \end{array} \qquad p_{R|D} = \begin{array}{c|cc} R\backslash D & 0 & 1 \\ \hline 0 & 1 & 0.1 \\ 1 & 0 & 0.9 \end{array}$$

(b) $p_{B,R}(0,1) = p_B(0)\sum_{i=0}^{1} p_{D|B}(i|0)p_{R|D}(1|i) = 0.05\cdot(0.1\cdot 0+0.9\cdot 0.9) = 0.0405$, and $p_{B,R}(1,1) = p_B(1)\sum_{i=0}^{1} p_{D|B}(i|1)p_{R|D}(1|i) = 0.95\cdot(0.99\cdot 0 + 0.01\cdot 0.9) = 0.00855$, so $p_R(1) = 0.0405 + 0.00855 = 0.04905$, and

$$p_{B|R}(0|1) = \frac{p_{B,R}(0,1)}{p_R(1)} \approx \underline{\underline{0.8257}}.$$

2. (a) Only the node $\underline{\underline{B}}$ is d-separated from F.

(b) $p_{B,C}(0,1) = \sum_{a=0}^{1} p_A(a)p_{B|A}(0|a)p_{C|A}(1|a) = 0.4\cdot 0.3\cdot 0.8+0.6\cdot 0.6\cdot 0.1 = 0.132$, and $p_B(0) = \sum_{a=0}^{1} p_A(a)p_{B|A}(0|a) = 0.4\cdot 0.3 + 0.6\cdot 0.6 = 0.48$, so we get:

$$p_{C|B}(1|0) = \frac{p_{B,C}(0,1)}{p_B(0)} = \underline{\underline{0.275}}.$$

(c) The "do"-conditioning removes the edge (A,B) from the DAG. In the new DAG, $B \perp C \parallel_{\mathcal{G}} \emptyset$, so B and C are independent. Therefore, $p_{C\|B}(1 \parallel 0) = p_C(1) = \sum_{a=0}^{1} p_A(a)p_{C|A}(1|a) = 0.4\cdot 0.8+0.6\cdot 0.1 = \underline{\underline{0.38}}$.

3. (a) The ML estimates are: $\widehat{\theta}_{A,1} = \underline{\underline{0.6}}$; $\widehat{\theta}_{B,1,1} = \underline{\underline{\frac{2}{3}}}$; $\widehat{\theta}_{E,0,1,0} = \underline{\underline{0.25}}$.

(b) The prior distributions are Beta(3, 2) and Beta(1, 1.5). Since

$$\frac{\Gamma(9+6)}{\Gamma(9)\Gamma(6)} = \frac{14!}{8!5!} = 18018; \qquad \frac{\Gamma(2+4.5)}{\Gamma(2)\Gamma(4.5)} = \frac{5.5 \cdot 4.5\Gamma(4.5)}{\Gamma(4.5)} = 24.75,$$

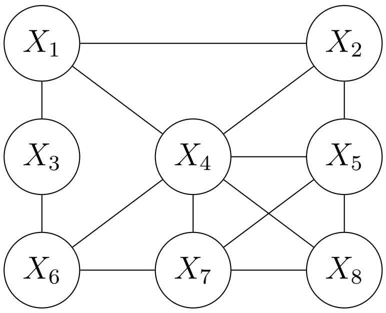
the posterior pdfs are:

$$f_{A,1}(\theta) = \begin{cases} 18018 \cdot \theta^8(1-\theta)^5, & 0 < \theta < 1; \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{E,0,1,0}(\theta) = \begin{cases} 24.75 \cdot \theta(1-\theta)^{3.5}, & 0 < \theta < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(c) The Bayes estimates are: $\widehat{\theta}_{A,1} = \frac{9}{15} = \underline{\underline{0.6}}$; $\widehat{\theta}_{E,0,1,0} = \frac{2}{6.5} \approx \underline{\underline{0.3077}}$.

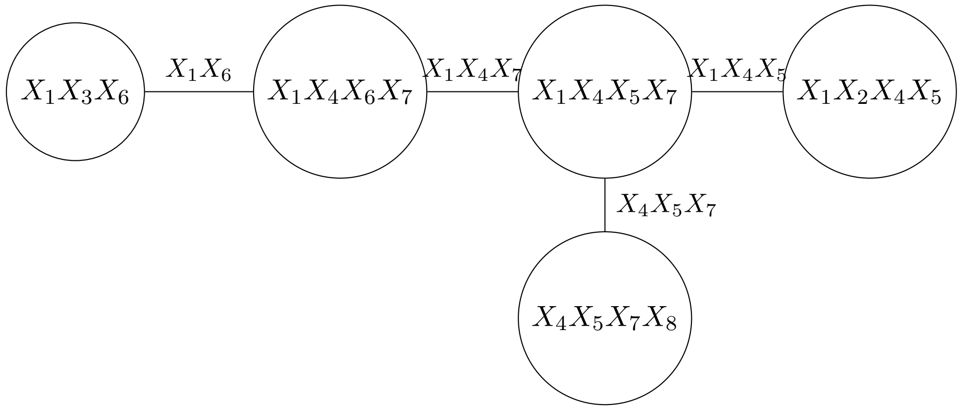
4. (a) The moralized graph:



The moralized graph $\underline{\underline{\text{is not}}}$ decomposable, since it contains a cycle of length 4 without a chord: $X_1 - X_4 - X_6 - X_3 - X_1$.

(b) X_8, X_2 (fill-in: $X_1 - X_5$), X_3 (fill-in: $X_1 - X_6$), X_6 (fill-in: $X_1 - X_7$), X_1, X_5, X_7, X_4.

(c) A junction tree:



A fully active schedule:

$X_1X_3X_6 \to X_1X_4X_6X_7$; $X_1X_4X_6X_7 \to X_1X_4X_5X_7$; $X_1X_2X_4X_5 \to X_1X_4X_5X_7$; $X_4X_5X_7X_8 \to X_1X_4X_5X_7$; $X_1X_4X_5X_7 \to X_4X_5X_7X_8$; $X_1X_4X_5X_7 \to X_1X_2X_4X_5$; $X_1X_4X_5X_7 \to X_1X_4X_6X_7$; $X_1X_4X_6X_7 \to X_1X_3X_6$.

5. (a) There are exactly three such subsets: $\{C\}$, $\{E\}$, and $\{C, E\}$.

   (b) According to the intervention formula,

$$p_{V\setminus\{D\}||D} = \frac{p_V}{p_{D|C}} = p_C p_{E|C} p_{A|E} p_{F|D} p_{B|A,F}.$$

   Marginalizing over B, F, E, and C (in that order), we get:

$$p_{A||D}(a||d) = \sum_{c\in\chi_C}\sum_{e\in\chi_E} p_C(c) p_{E|C}(e|c) p_{A|E}(a|e) \qquad \forall a \in \chi_A, d \in \chi_D.$$

   Since $A \perp C \parallel_{\mathcal{G}} E$ (why?), it holds that $A \perp C|E$, so $p_{A|C,E}(a|c,e) = p_{A|E}(a|e)$ for all $a \in \chi_A, c \in \chi_C, e \in \chi_E$ such that $p_{C,E}(c,e) > 0$. Therefore,

$$p_{A||D}(a||d) = \sum_{c\in\chi_C}\sum_{e\in\chi_E} p_C(c) p_{E|C}(e|c) p_{A|C,E}(a|c,e)$$

$$= \sum_{c\in\chi_C}\sum_{e\in\chi_E} p_C(c) p_{A,E|C}(a,e|c) = \sum_{c\in\chi_C} p_C(c) p_{A|C}(a|c) \quad \forall a \in \chi_A, d \in \chi_D.$$

6. We must show that $X \perp Y \parallel_{\mathcal{G}} Z$ and that $X \perp W \parallel_{\mathcal{G}} Z$. To prove the first claim, assume that τ is a trail in $\mathcal{G}$ connecting X and Y, such that no node in τ belongs to Y (apart from the end node). Then, either τ has a collider node α such that neither α nor any of its descendants

belong to $W \cup Z$, in which case Z blocks τ; or τ has no such collider node, but has a chain or fork node β which belongs to $W \cup Z$. If $\beta \in Z$, then Z blocks τ, while if $\beta \in W$ (and τ has no chain or fork node belonging to Z), then there is a trail $\tau' \subset \tau$ (a subtrail of τ) connecting X and W, which contains no chain or fork node belonging to $Y \cup Z$. By assumption, then, τ' has a collider node α' such that neither α' nor any of its descendants belong to $Y \cup Z$, implying that Z blocks τ.

The second claim is proven in exactly the same way, by reversing the roles of Y and W.