1.11 The r.v.s $X_1, X_2, \dots, X_n, X_{n+1}$ have the same state space: $\mathcal{X}_1 = \{x^{(1)}, x^{(2)}, \dots, x^{(k)}\}$. They are conditionally independent given $\Theta=\theta$, and all have the same conditional pmf given $\Theta=\theta$, namely: $p_{X_i|\Theta}(x^{(j)}|\theta) = \theta_j$, $j=1,2,\dots,k$. Let $\bar{X}_n = (X_1, \dots, X_n)^T$.

The generalized law of total expectation says that, for any discrete r.v.s (X, Y) and any continuous ($k-1$ dim.) r.v. Z:

$$p_{X|Y}(x|y) = \int_{\mathbb{R}^{k-1}} p_{X|Y,Z}(x|y,z) f_{Z|Y}(z|y)\, dz$$

If we choose $X = X_{n+1}$, $Y = \bar{X}_n$, and $Z = \Theta = (\Theta_1, \dots, \Theta_{k-1})^T$, we get:

$$p_{X_{n+1}|\bar{X}_n}(x_{n+1}|\bar{x}_n) = \int_{\mathbb{R}^{k-1}} p_{X_{n+1}|\bar{X}_n,\Theta}(x_{n+1}|\bar{x}_n,\theta) f_{\Theta|\bar{X}_n}(\theta|\bar{x}_n)\, d\theta.$$

Since $X_{n+1} \perp \bar{X}_n | \Theta$, we get: $p_{X_{n+1}|\bar{X}_n,\Theta}(x_{n+1}|\bar{x}_n,\theta) = p_{X_{n+1}|\Theta}(x_{n+1}|\theta) =$

$$= \begin{cases} \theta_j, & \text{if } x_{n+1} = x^{(j)}, \ j=1,\dots,k. \\ 0, & \text{otherwize} \end{cases}$$

Combining these results, we get: $\boxed{p_{X_{n+1}|\bar{X}_n}(x^{(j)}|\bar{x}_n) = \int_{\mathbb{R}^{k-1}} \theta_j f_{\Theta|\bar{X}_n}(\theta|\bar{x}_n)\, d\theta.}$

Now, $f_{\Theta|\bar{X}_n}(\theta|\bar{x}_n) = \dfrac{\Gamma(n+\alpha)}{\prod_{j=1}^{k}\Gamma(n_j+\alpha_j)} \prod_{i=1}^{k} \theta_i^{n_i+\alpha_i-1}$, $0 \le \theta_i \le 1$, $i=1,\dots,k-1$, $\sum_{i=1}^{k-1}\theta_i < 1$.

which is the pdf of the Dirichlet$(\alpha_1+n_1, \dots, \alpha_k+n_k)$. Therefore,

$\boxed{p_{X_{n+1}|\bar{X}_n}(x^{(j)}|\bar{x}_n) = E(\Theta_j)}$, where $\Theta = (\Theta_1, \dots, \Theta_{k-1})^T \sim$ Dirichlet$(\alpha_1+n_1, \dots, \alpha_k+n_k)$.

Can you compute $E(\Theta_j)$?

Can you compute $V(\Theta_j)$? (This is what you are asked to do in problem 1.12.)