TAMS22/TEN1 SANNOLIKHETSTEORI OCH BAYESIANSKA NÄTVERK
TENTAMEN MÅNDAG 10 JANUARI 2022 KL 14.00-18.00.

Examinator och jourhavande lärare: Torkel Erhardsson, tel. 28 14 78.

Permitted exam aids: A calculator with empty memories. One sheet of paper (format A4) with the student's handwriting on both sides.

The exam consists of 6 problems worth 3 points each. Grading limits : 8 points for grade 3, 11.5 points for grade 4, 15 points for grade 5. The results will be communicated by email.

Problem 1

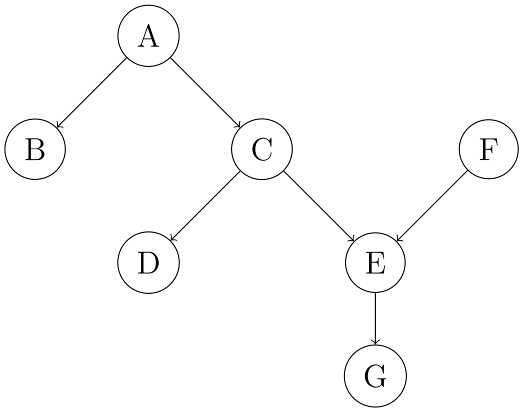
Jason, the robot juggler, drops the ball quite often when the battery is low. In previous trials, it has been determined that when the battery is low, he drops the ball 9 times out of 10. When the battery is not low, the chance that he drops the ball is just 1%. The battery was recharged recently, so there is only a 5% chance that the battery is low. Another robot, Olga the observer, reports on whether or not Jason has dropped the ball. Unfortunately, Olga's vision is somewhat unreliable. When Jason drops the ball, there is a 10% chance that Olga will not detect it. You may assume that Olga never detects a dropped ball if Jason doesn't drop it.

(a) Construct a Bayesian network to represent the situation described above, and write down the corresponding probability tables.

(b) Olga reports that Jason has dropped the ball. Given this evidence, what is the (updated) probability that the battery is low?

Problem 2

The binary valued random variables $\{A, B, C, D, E, F, G\}$ have a joint pmf which factorizes along the following DAG:



The corresponding conditional probability tables are: $p_A(1) = 0.6$, $P_F(1) = 0.8$,

$$p_{B|A} = \begin{array}{c|cc} B\backslash A & 0 & 1 \\ \hline 0 & 0.3 & 0.6 \\ 1 & 0.7 & 0.4 \end{array} \qquad p_{C|A} = \begin{array}{c|cc} C\backslash A & 0 & 1 \\ \hline 0 & 0.2 & 0.9 \\ 1 & 0.8 & 0.1 \end{array}$$

$$p_{D|C} = \begin{array}{c|cc} D\backslash C & 0 & 1 \\ \hline 0 & 0.1 & 0.8 \\ 1 & 0.9 & 0.2 \end{array} \qquad p_{G|E} = \begin{array}{c|cc} G\backslash E & 0 & 1 \\ \hline 0 & 0.2 & 0.3 \\ 1 & 0.8 & 0.7 \end{array}$$

$$p_{E|C,F} = \begin{array}{c|cccc} E\backslash(C,F) & (0,0) & (0,1) & (1,0) & (1,1) \\ \hline 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 1 & 0.9 & 0.8 & 0.7 & 0.6 \end{array}$$

(a) Which nodes (if any) are d-separated from F by the set $\{A, G\}$?

(b) Compute $p_{C|B}(1|0)$.

(c) Compute the "do"-conditional probability $p_{C\|B}(1 \parallel 0)$.

Problem 3

Consider again the DAG in Problem 2. The following table contains 10 independent observations of $(A, B, C, D, E, F, G)^T$.

	$u^{(1)}$	$u^{(2)}$	$u^{(3)}$	$u^{(4)}$	$u^{(5)}$	$u^{(6)}$	$u^{(7)}$	$u^{(8)}$	$u^{(9)}$	$u^{(10)}$
A	1	1	1	0	0	0	0	1	1	1
B	1	1	1	1	1	0	0	0	0	1
C	0	0	1	0	0	1	1	1	1	0
D	0	0	1	0	1	1	0	1	0	0
E	1	1	1	0	0	1	1	0	0	1
F	0	1	0	1	1	0	0	1	0	0
G	1	0	1	0	1	1	1	1	0	1

Define $\theta_{A,1} = p_A(1)$, $\theta_{B,1,1} = p_{B|A}(1|1)$, and $\theta_{E,0,1,0} = p_{E|C,F}(0|1,0)$.

(a) Compute the ML estimates of $\theta_{A,1}$, $\theta_{B,1,1}$, and $\theta_{E,0,1,0}$, based on the observations.

(b) Assume that $\theta_{A,1}$ and $\theta_{E,0,1,0}$ are given (independent) prior distributions, with pdfs

$$f_{A,1}(\theta) = \begin{cases} 12\theta^2(1-\theta), & 0 < \theta < 1; \\ 0, & \text{otherwise,} \end{cases}$$

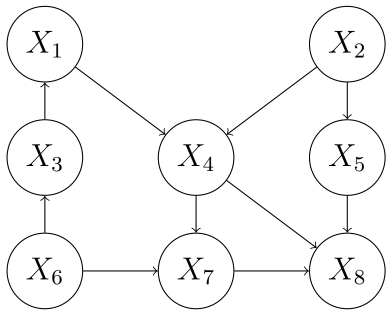
$$f_{E,0,1,0}(\theta) = \begin{cases} 1.5\sqrt{1-\theta}, & 0 < \theta < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Compute the pdfs of the posterior distributions of $\theta_{A,1}$ and $\theta_{E,0,1,0}$ given the observations. Remember that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

(c) Using the same prior distributions as in (b), compute the Bayes estimates of $\theta_{A,1}$ and $\theta_{E,0,1,0}$ given the observations.

Problem 4

Let $V = \{X_1, \ldots, X_8\}$ be binary valued random variables, whose joint pmf $p_{X_1,\ldots,X_8}$ factorizes along the DAG $\mathcal{G} = (V, E)$:



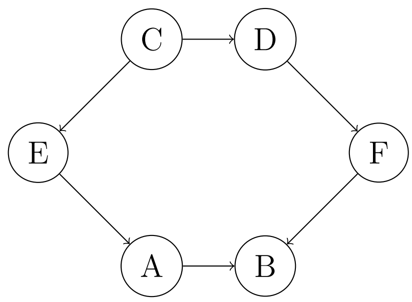
(a) Moralize $\mathcal{G}$. Is the moralized graph $\mathcal{G}^m$ is decomposable? The answer must be supported by an argument.

(b) For the moralized graph $\mathcal{G}^m$, write down an elimination sequence which at each step produces as few fill-ins as possible. What (if any) are the fill-ins required for this elimination sequence?

(c) Organize the nodes of the triangulated graph into a junction tree, and write down a fully active schedule for local message passing.

Problem 5

Let $V = \{A, B, C, D, E, F\}$ be random variables with finite state spaces $\chi_A, \chi_B, \ldots, \chi_F$, whose joint pmf factorizes along the DAG $\mathcal{G} = (V, E)$:



(a) Find all subsets of V which satisfy the back door criterion with respect to the ordered pair (D, A).

(b) Use the intervention formula to prove that

$$p_{A||D}(a||d) = \sum_{c \in \chi_C} p_C(c) p_{A|C}(a|c) \qquad \forall a \in \chi_A, d \in \chi_D.$$

Problem 6

Let $\mathcal{G} = (V, E)$ be a DAG, and let $X, Y, Z, W \subset V$ denote four disjoint subsets of the node set V. Prove that if $X \perp Y \parallel_{\mathcal{G}} W \cup Z$ and $X \perp W \parallel_{\mathcal{G}} Y \cup Z$, then it holds that $X \perp (Y \cup W) \parallel_{\mathcal{G}} Z$.