Solutions

TAMS24/TEN1 2019-08-23

1. (a) We know that

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

so since σ is known, we could use $\overline{X}$ directly. However, we might as well follow the usual procedure. If H_0 holds, then

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1).$$

The critical region C is chosen so that

$$P(Z \in C \mid H_0) = \alpha,$$

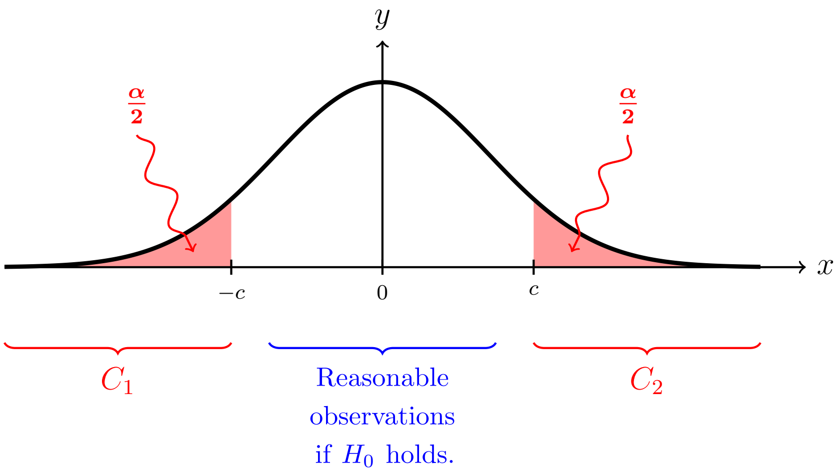
where we due to symmetry assume the form

$$C = \{z \in \mathbf{R} : |z| > c\} = \{z \in \mathbf{R} : z > c \text{ or } z < -c\}.$$

We note that C consists of two parts C_1 and C_2, where C_1 is on the negative half-axis. Due to symmetry,

$$P(Z \in C_1) = P(Z \in C_2) = \frac{\alpha}{2}.$$

In our case, $\alpha = 0.05$, so $c = \Phi^{-1}(0.975) = 1.96$.



Using our observations, we find that the test statistic is given by

$$z = \frac{\overline{x} - 3.0}{\sqrt{0.0625}/\sqrt{8}} = 1.7118 \notin C$$

so we can not reject H_0.

(b) The power at $\mu = 3.1$ can be calculated straight from the definition. Remember that $Z = \dfrac{\overline{X} - 3.0}{\sqrt{0.0625}/\sqrt{8}}$, so if $\mu = 3.1$, then

$$Z \sim N\left(\frac{0.1}{\sqrt{0.0625}/\sqrt{8}}, 1\right) = N(1.1314, 1).$$

Hence,

$$h(3.1) = P(H_0 \text{ rejected} \,|\, \mu = 3.1) = P(Z \in C \,|\, \mu = 3.1) = P\left(Z < -1.96 \text{ eller } Z > 1.96\right)$$
$$= \Phi(-1.96 - 1.1314) + 1 - \Phi(1.96 - 1.1314) = 0.2047.$$

(c) Given the observation $\overline{x} = 3.1513$ and $z = 1.7118$, we can follow the same procedure as in the previous exercise. However, in this case with C unknown. It is clear that we want to choose $c = 1.7118$ and since we found $c = \Phi^{-1}(1 - \alpha/2)$, it follows that

$$1.7118 = \Phi^{-1}(1 - \alpha/2) \quad \Leftrightarrow \quad \Phi(1.7118) = 1 - \frac{\alpha}{2},$$

so $\alpha = 2(1 - \Phi(1.7118)) = 0.087$. So we can choose the significance level 8.7% and still reject H_0. This is not good in practice! You should not look at the data to choose your significance level.

Answer: (a) We can not reject H_0. (b) 0.205 (c) $\alpha = 0.087$.

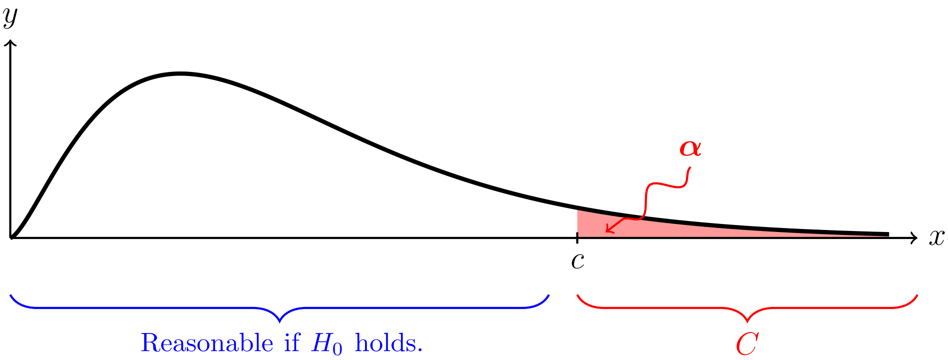
2. (a) Let $H_0 : \sigma^2 = 0.0625$ and the alternate hypothesis be $H_1 : \sigma^2 > 0.0625$. We have $n = 8$ samples, so

$$V = \frac{7\, S^2}{0.0625} \sim \chi^2(7).$$

We're seeking a limit c so that

$$\alpha = P(V > c) = P\left(S^2 > \frac{c\sigma_0^2}{n-1}\right)$$

and define the critical region as $C =]c, \infty[$.



From a table, we find $c = 14.07$, so

$$v = \frac{7 \cdot s^2}{0.0625} = 17.39 \in C.$$

We therefore reject H_0 and claim that it is very likely that the variance is greater than 0.0625.

(b) It follows that (by Cochran's and Gosset's theorems)

$$T_X = \frac{\overline{X} - \mu_X}{S/\sqrt{8}} \sim t(7),$$

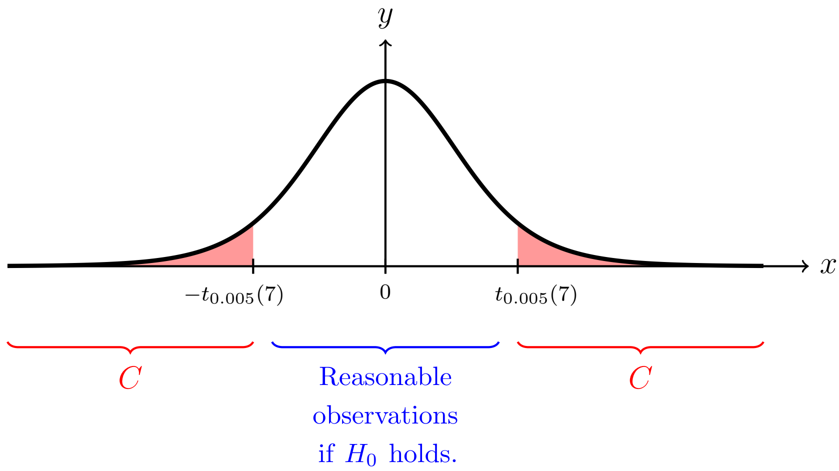
and

$$P(-t_{\alpha/2}(7) < T_X < t_{\alpha/2}(7)) = 1 - \alpha,$$

where we can solve the inequality for

$$\overline{X} - t_{\alpha/2}(7) \cdot \frac{S}{\sqrt{8}} < \mu_X < \overline{X} + t_{\alpha/2}(7) \cdot \frac{S}{\sqrt{8}}.$$

From a table, we find that $t_{0.005}(7) = 3.50$.



As an observation of S_X, we use $\sqrt{s_X^2}$, so

$$t_{0.005}(7)\frac{s}{\sqrt{8}} = 3.50 \cdot \frac{0.3943}{2.8284} = 0.4879.$$

Since $\overline{x} = 3.1513$, the interval is given by

$$I_{\mu_X} = (2.66,\ 3.64).$$

Answer: (a) We reject H_0. We believe that the variance is higher. (b) (2.66, 3.64).

3. We need to organize the data so that we can see how many machines were taken out of action in each time interval. We also need to choose these intervals so that — under the assumption that times are Exp($\mu = 1.0$)-distributed — all intervals are comparable in probability. The rule of thumb is to use $50/10 = 5$ classes, so we can try the following.

Time	How many
$I_1 = [0,\ 6)$	11
$I_2 = [6,\ 12)$	8
$I_3 = [12,\ 24)$	12
$I_4 = [24,\ 36)$	8
$I_5 = [36,\ \infty)$	11

It is not clear that this partitioning is good enough. Let H_0 be the hypothesis that times are Exp($\mu = 1.0$)-distributed. If H_0 is true, then the probability density of the time before a machine has to have its blades sharpened is given by $f(x) = \mu^{-1}\exp(-\mu^{-1}x)$ (with $\mu = 1.0$), so

$$P(a \le X < b) = \int_a^b \frac{1}{\mu}\exp\left(-\frac{x}{\mu}\right)dx = \exp\left(-\frac{a}{\mu}\right) - \exp\left(-\frac{b}{\mu}\right).$$

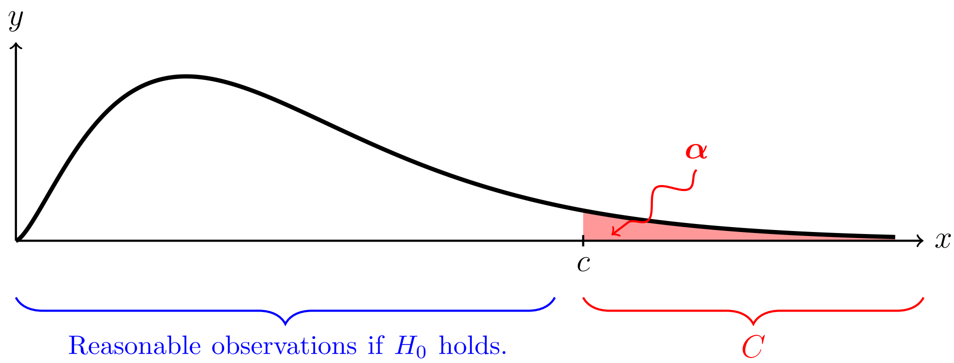
With these numbers, we can do the calculations and see that the probability of ending up in each interval is given by

$$P(X \in I_k) = \begin{cases} p_1 = 0.2212, & k = 1, \\ p_2 = 0.1723, & k = 2, \\ p_3 = 0.2387, & k = 3, \\ p_4 = 0.1447, & k = 4, \\ p_5 = 0.2231, & k = 5. \end{cases}$$

The testing quantity we now use is given by

$$q = \sum_{j=1}^{5} \frac{(x_j - np_j)^2}{np_j} = \frac{(11 - 50 \cdot 0.2212)^2}{50 \cdot 0.2212} + \cdots + \frac{(11 - 50 \cdot 0.2231)^2}{50 \cdot 0.2231} = 8.65.$$

If H_0 is true, then q is an observation of $Q \overset{\text{appr.}}{\sim} \chi^2(5-1) = \chi^2(4)$. We reject H_0 if q is large, so we need a critical region C of the form $C = [c, \infty)$. From a table we find that $c = \chi^2_{0.10}(4) = 7.78$. If $q \geq c$, we reject H_0.



Since $q \in C$, the conclusion is that we reject H_0. We do not believe that *Pleasures of the flesh* is telling the truth.

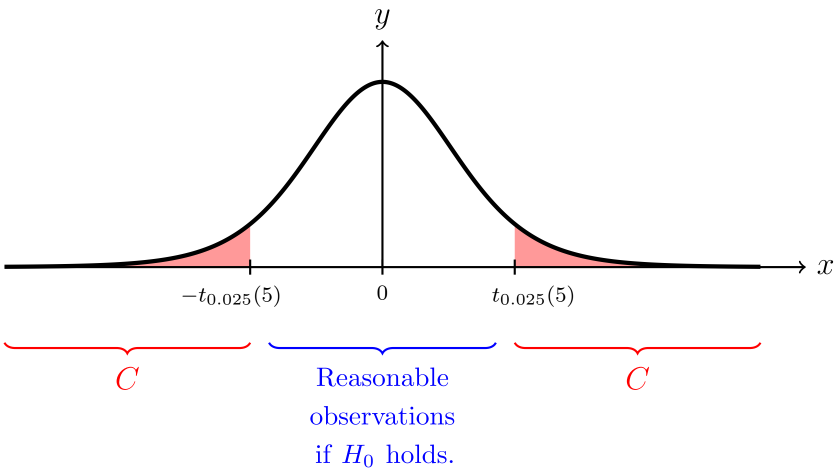
Answer: We reject the assumption.

4. (a) We can perform this test in several different ways. We can test whether $\beta_2 = 0$ in model 2 directly or we can compare model 1 and model 2 and see if model 2 is significantly better.

   Alternative 1. To test if $\beta_2 = 0$, let $H_0 : \beta_2 = 0$ and $H_1 : \beta_2 \neq 0$. Assume that H_0 holds. Then

$$T = \frac{\widehat{\beta}_2 - 0}{S\sqrt{h_{22}}} \sim t(8-3) = t(5),$$

   where the distribution is clear since H_0 holds. We need a critical region C such that $P(T \in C \mid H_0) = 0.05$ and since H_1 is double sided, we choose symmetrically.



We find $t_{\alpha/2}(5) = t_{0.025}(5) = 2.57061$ in a table. An observation of $S\sqrt{h_{22}}$ is given by the standard error $d(\widehat{\beta}_2)$ and thus we find that the observation

$$t = \frac{1.0324}{0.2876} = 3.59$$

does belong to the critical region. So we reject H_0. The coefficient β_2 is probably not zero.

Alternative 2.

We have model 1:

$$y = \beta_0 + \beta_1 x_1 + \epsilon$$

and model 2:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon,$$

where $x_2 = x_1^2$. We can test if the second model is significantly better by testing whether $\beta_2 = 0$ in a slightly different way.

Let

$$H_0 : \beta_2 = 0,$$

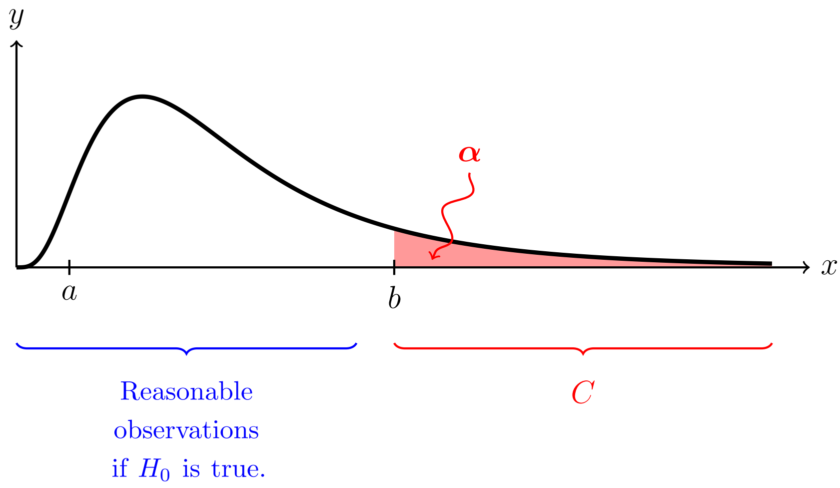
and

$$H_1 : \beta_2 \neq 0.$$

If H_0 is true, then $\boldsymbol{Y} \sim N(X_1\boldsymbol{\beta_1}, \sigma^2 I)$, so

$$W = \frac{(\mathrm{SS}_\mathrm{E}^{(1)} - \mathrm{SS}_\mathrm{E}^{(2)})/1}{\mathrm{SS}_\mathrm{E}^{(2)}/5} \sim F(1,5) \quad \text{if } H_0 \text{ is true}$$

since this is a quotient of independent χ^2 variables. If H_0 is not true, then W will tend to grow large. The critical region is given by $C =]c, \infty[$ for some $c > 0$.



From the table we find that $c = 10.0070$. An observation of W is found in

$$w = \frac{(0.0607 - 0.0170)/1}{0.0170/5} = 12.85,$$

so $w \in C$. We can reject the null hypothesis.

(b) We wish to find a confidence interval for β_1 using model 2. We know that

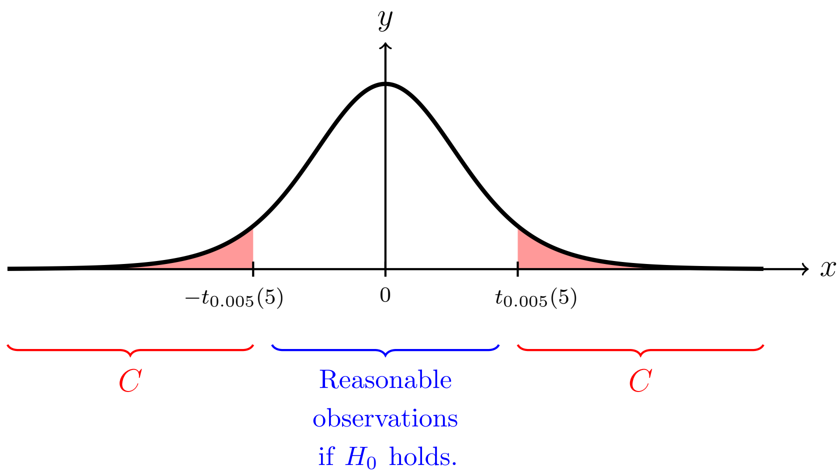
$$T = \frac{\widehat{\beta}_1 - \beta_1}{S\sqrt{h_{11}}} \sim t(5).$$

So

$$P(-t_{\alpha/2}(5) < T < t_{\alpha/2}(5)) = 1 - \alpha,$$

where we can solve the inequality for

$$\widehat{\beta}_1 - t_{\alpha/2}(5) \cdot S\sqrt{h_{11}} < \beta_1 < \widehat{\beta}_1 + t_{\alpha/2}(5) \cdot S\sqrt{h_{11}}.$$



From a table, we find that $t_{0.005}(4) = 4.0321$. An observation of $S\sqrt{h_{11}}$ is given by the standard error $d(\widehat{\beta}_1) = 0.3315$ and thus we find the confidence interval

$$I_{\beta_1} = \left(\widehat{\beta}_1 - 4.0321 \cdot 0.3315,\ \widehat{\beta}_1 + 4.0321 \cdot 0.3315\right) = (-0.37,\ 2.30).$$

A similar calculation for model 1 yields

$$I_{\beta_1} = \left(\widehat{\beta}_1 - 4.0321 \cdot 0.1241,\ \widehat{\beta}_1 + 4.0321 \cdot 0.1241\right) = (1.62,\ 2.63).$$

Different intervals with different interpretations. For model 2, we could perform a hypothesis test to investigate whether $\beta_1 = 0$ or not and the conclusion would be that $\beta_1 = 0$ might very well be true. For model 1, the conclusion is that $\beta_1 \neq 0$.

Answer:

(a) A significance test shows that we conclude that $\beta_2 \neq 0$ at the significance level 5%.

(b) $(-0.37,\ 2.30)$. Different intervals with different interpretations. See above.

5. (a) Let X_1 be the number of fish eaten in the first tank and X_2 the number of fish eaten in the second tank. Assuming independence, it is clear that $X_1 \sim \mathrm{Bin}(300, p_1)$ and that $X_2 \sim \mathrm{Bin}(300, p_2)$. As estimators we choose

$$\widehat{P_1} = \frac{X_1}{300}, \quad \widehat{P_2} = \frac{X_2}{300}, \quad \text{and} \quad \widehat{P_2 - P_1} = \widehat{P_2} - \widehat{P_1}.$$

We note that $E(\widehat{P_1}) = E(X_1)/300 = 300p_1/300 = p_1$ and similarly $E(\widehat{P_2}) = p_2$, so $E(\widehat{P_2 - P_1}) = p_2 - p_1$. Our estimators are unbiased.

Now, we have observed that $\widehat{p_1} = 100/300 = 1/3$ and that $\widehat{p_2} = 120/300 = 2/5$. The binomial distribution is a bit messy to deal with in this instance (discrete intervals?), so let's try an approximation instead. Since

$$300 \cdot \widehat{p_1} \cdot (1 - \widehat{p_1}) = 300 \cdot \frac{1}{3} \cdot \frac{2}{3} = 66.67$$

and

$$300 \cdot \widehat{p_2} \cdot (1 - \widehat{p_2}) = 300 \cdot \frac{2}{5} \cdot \frac{3}{5} = 72$$

are both greater than 10, a normal approximation is reasonable. Hence,

$$\widehat{P_1} \overset{\text{appr.}}{\sim} \mathrm{N}(p_1, p_1(1 - p_1)/300)$$

and

$$\widehat{P_2} \overset{\text{appr.}}{\sim} \mathrm{N}(p_2, p_2(1 - p_2)/300).$$

From this it also follows that

$$\widehat{P_2} - \widehat{P_1} \overset{\text{appr.}}{\sim} \mathrm{N}(p_2 - p_1, p_1(1 - p_1)/300) + p_2(1 - p_2)/300).$$

We now have

$$Z_1 = \frac{\widehat{P_1} - p_1}{\sqrt{\widehat{p_1}(1 - \widehat{p_1})/300}} \overset{\text{appr.}}{\sim} N(0, 1).$$

Note that we've replaced p_1 by the estimate $\widehat{p_1}$ in the denominator. Similarly

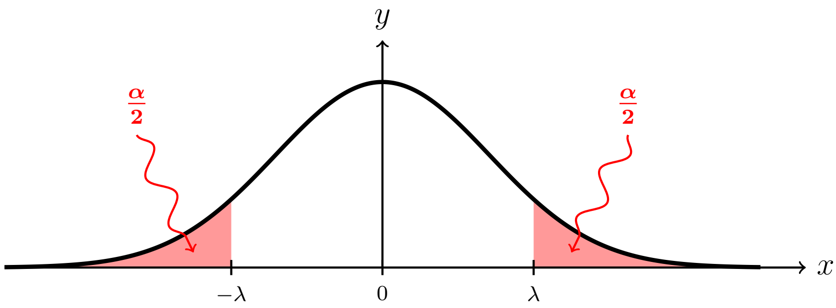
$$Z_2 = \frac{\widehat{P_2} - p_2}{\sqrt{\widehat{p_2}(1 - \widehat{p_2})/300}} \overset{\text{appr.}}{\sim} N(0, 1)$$

and

$$Z = \frac{\widehat{P_2} - \widehat{P_1} - (p_2 - p_1)}{\sqrt{\widehat{p_2}(1 - \widehat{p_2})/300 + \widehat{p_1}(1 - \widehat{p_1})/300}} \overset{\text{appr.}}{\sim} N(0, 1).$$

One reason for approximating the denominator is that we can use the normal distribution directly (with known variance). We seek a number λ so that, e.g.,

$$P(-\lambda < Z < \lambda) = 0.95.$$



We find $\lambda = 1.96$ from a table ($\lambda = \Phi^{-1}(0.975)$). So approximate confidence intervals (with 95% degree of confidence) can be found in

$$I_{p_1} = (0.333 - 1.96 \cdot 0.0272,\ 0.333 + 1.96 \cdot 0.0272) = (0.28, 0.39),$$

$$I_{p_2} = (0.4 - 1.96 \cdot 0.0283,\ 0.4 + 1.96 \cdot 0.0283) = (0.34, 0.46),$$

and since

$$\sqrt{\widehat{p_1}(1-\widehat{p_1})/300 + \widehat{p_2}(1-\widehat{p_2})/300} = 0.0393,$$

we obtain

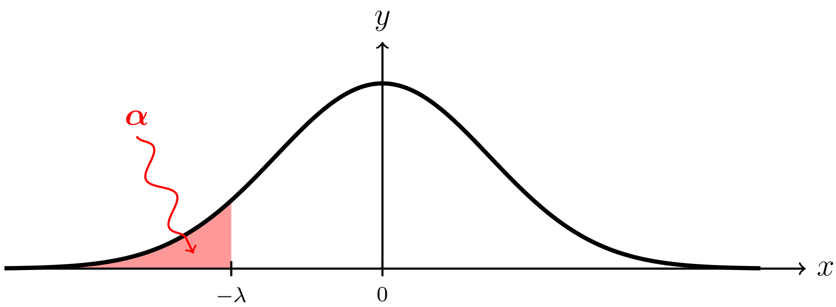
$$I_{p_2-p_1} = (0.0667 - 1.96 \cdot 0.0393,\ 0.0667 + 1.96 \cdot 0.0393) = (-0.01, 0.15).$$

Since $0 \in I_{p_2-p_1}$, we can't reject the hypothesis that $p_1 = p_2$.

(b) In this case, we want to test against the alternate hypothesis that $p_2 > p_1$ (not that $p_1 \neq p_2$). We use the same significance level, but place all the uncertainty in one tail. We use the same estimator for $p_2 - p_1$ and transform according to

$$Z = \frac{\widehat{P_2} - \widehat{P_1} - (p_2 - p_1)}{\sqrt{\widehat{p_2}(1-\widehat{p_2})/300 + \widehat{p_1}(1-\widehat{p_1})/300}} \overset{\text{appr.}}{\sim} N(0,1).$$

We seek λ so that $P(Z > -\lambda)$.



Since all the probability α is in the left tail, this pushes λ closer to the origin. From a table we find that $\lambda = \Phi^{-1}(0.95) = 1.645$. Then

$$-\lambda < Z \quad \Leftrightarrow \quad \widehat{P_2} - \widehat{P_1} - \lambda\sqrt{\widehat{p_2}(1-\widehat{p_2})/300 + \widehat{p_1}(1-\widehat{p_1})/300} < p_2 - p_1.$$

Using the point estimates for $\widehat{P_1}$ and $\widehat{P_2}$, we find that

$$p_2 - p_1 > 0.0021.$$

Hence $I_{p_2-p_1} = (0.0021, 1)$. We can now reject the hypothesis that $p_2 = p_1$ and claim that $p_2 > p_1$ is very likely.

Answer: (a) $I_{p_1} = (0.28, 0.39)$, $I_{p_2} = (0.34, 0.46)$ and $I_{p_2-p_1} = (-0.01, 0.15)$. Nope. (b) $I_{p_2-p_1} = (0.0021, 1)$. The second type is likely more ferocious.

6. We note that the covariance matrix can be written more compactly as $\boldsymbol{C} = I - \boldsymbol{q}\boldsymbol{q}^T$, where $\boldsymbol{q} = (\sqrt{p_1}\ \sqrt{p_2}\ \cdots\ \sqrt{p_k})^T$. Using this representation, we can verify that

$$(I - \boldsymbol{q}\boldsymbol{q}^T)^2 = I - \boldsymbol{q}\boldsymbol{q}^T \quad \text{and} \quad (I - \boldsymbol{q}\boldsymbol{q}^T)^T = I - \boldsymbol{q}\boldsymbol{q}^T,$$

so $\boldsymbol{C} = I - \boldsymbol{q}\boldsymbol{q}^T$ is a projection matrix and therefore has the eigenvalues $\lambda = 0$ and $\lambda = 1$. For these types of matrices, we know that the rank is equal to the trace. Since the trace of the matrix is equal to the sum of the eigenvalues, it is clear that

$$\operatorname{rank}(I - \boldsymbol{p}\boldsymbol{p}^T) = \operatorname{tr}(I - \boldsymbol{p}\boldsymbol{p}^T) = k - (p_1 + p_2 + \cdots + p_n) = k - 1,$$

so $\lambda = 0$ is a simple eigenvalue. The matrix is symmetric and positive semidefinite, so there exists an orthonormal matrix U such that $U^T\boldsymbol{C}U = \operatorname{diag}(1, 1, \ldots, 1, 0)$ becomes a diagonal matrix. If we let $\boldsymbol{Z} = U\boldsymbol{Y}$, we see that $\boldsymbol{Z} \sim N(\boldsymbol{0}, \operatorname{diag}(1, 1, \ldots, 1, 0))$ and that

$$\boldsymbol{Y}^T\boldsymbol{Y} = (U^T\boldsymbol{Z})^T U^T\boldsymbol{Z} = \boldsymbol{Z}^T\boldsymbol{Z} = \sum_{j=1}^{k-1} Z_j^2,$$

where $Z_j \sim N(0, 1)$ are independent. This sum is obviously $\chi^2(k-1)$-distributed!

Answer: See above.