LINKÖPINGS UNIVERSITET

Matematiska institutionen

Matematisk statistik

Utbildningskod: TAMS24

Modul: TEN1

Datum: 2019-09-07

Institution: MAI

Exam in Statistics
TAMS24/TEN1 2019-09-07

You are permitted to bring:

- a calculator (no computer);

- Formel- och tabellsamling i matematisk statistik (from MAI);

- Formel- och tabellsamling i matematisk statistik, TAMS65;

- TAMS24: Notations and Formulas (by Xiangfeng Yang).

- At the mountains of madness (H.P. Lovecraft).

Grading (sufficient limits): 8-11 points giving grade 3; 11.5-14.5 points giving grade 4; 15-18 points giving grade 5. Your solutions need to be complete, well motivated, carefully written and concluded by a clear answer. Be careful to show what is random and what is not. Assumptions you make need to be explicit. Approximations are allowed if reasonable and clearly motivated. The exercises are in random order.

Solutions can be found on the homepage a couple of hours after the finished exam.

ONCE upon a time, in a land far far away, Rick and Gary had run into problems with their flamingo farm Exodus. Someone had posted threatening notes all around the farm. Gary suggested that it was the animal liberation army (or was it perhaps the different organization known as the liberation army of the animals?) that disliked the fact that there was flamingos in captivity (and, perhaps, also the slaughtering of said animals). Rick — on the other hand — assumed that it was the cult (devoted to the crawling chaos) that usually drifted around in the wooden area mumbling on about the great old ones and the unmentionable horrors at the mountains of madness. In either case, the result was that Exodus was shut down and Gary and Rick went into hiding. When they finally managed to return, things had taken a turn for the worse. The beautiful exodus sign had been scribbled over with frantic writing in something brownish red, stating that *Beneath the Columns of Abandoned Gods lies Dormant Hallucinations, where the Conjuration of the Sepulchral results in The Sleep of Morbid Dreams. In The Dead of Winter, Pestilential Winds causes the Exhumation of the Ancient.*
The air felt stale and suddenly there was no wind. Slowly, they entered the compound.

1. Since the flamingos had been left to their own devices while Exodus was shut down, they had managed to get into the storage where all the growth hormone was kept. By some coincidence — or perhaps supernatural reason caused by colors out of space — some of the flamingos had managed to ingest huge amounts of hormone and developed rapidly. Taking a random sample of the surviving flamingos (apparently aggression levels had gone up causing quite a lot of conflict), Gary wants to investigate the current state of affairs. Gary's measurements can be seen below.

$$7.18 \quad 7.69 \quad 6.09 \quad 7.31 \quad 7.09 \quad 6.46 \quad 6.80 \quad 7.10$$

Assume that the samples are independent and from a normal distribution with unknown variance σ^2 and an unknown expectation μ_{now}.

(a) Find a confidence interval (99% degree of confidence) for the expectation μ_{now}. (1p)

(b) Gary also found his old notes from before where he obtained the following random sample of weights.

$$2.69 \quad 2.90 \quad 3.23 \quad 3.52 \quad 2.65 \quad 3.71 \quad 3.46 \quad 3.05$$

Assume that these two samples are independent and that the old ones are from an $N(\mu_{\text{old}}, \sigma^2)$-distribution. Test the hypothesis that the expected weight before is less than half of the current expected weight at the significance level 5%. (2p)

(c) Test the assumption that the variance of the two previous samples is the same. Use the significance level 5%. Conclusion? (2p)

2. After Gary shared his findings, Rick came clean about a mistake he made before the shut down. A New Mexico-native woman called Trinity had sold him some beautiful sand, full of greenish glass-like particles, that she had brought with her from the desert near Alamogordo. Rick had poured out several hundreds of kilos all around the water pond where the flamingos congregated. Unfortunately, it turned out that the sand contained high amounts of plutonium and fission products thereof. Rick is worried that the radioactivity is affecting the flamingos, so he takes some measurements using his old but trustworthy Geiger counter. The counter is calibrated to measure Giga Becquerel (1 Bq means 1 decay per second). Assume that the decay can be characterized by a Poisson process $X(t)$ with intensity $\lambda > 0$. Assume that Rick's counter reading is an observation of an $X = X(1) \sim \text{Po}(\mu)$ variable. When Rick took his measurement he obtained $x = 8$. For some reason, Rick felt that if the expectation μ wasn't greater than 5, everything was good enough (Not great. Not terrible).

Let $H_0 : \mu = 5$ and $H_1 : \mu > 5$.

(a) Perform a hypothesis test using the null hypothesis H_0 and the alternate hypothesis H_1 at the significance level 5%. (2p)

(b) What is the power of the test at $\mu = 10$ (1p)

3. Assuming the same situation as in the previous exercise, additionally assume that we measure for 10 seconds and obtain the observation $y = 82$ of the random variable $Y = X(10)$ (meaning that we measure the stochastic process $X(t)$ for 10 seconds; $t = 10$). Find a 90% confidence interval for the intensity λ. (2p)

4. During their absence, cult members had obviously gained access to the farm and carried out their vile and unspeakable rituals. Both Gary and Rick found that the place felt very different from before. Shadows were angling in weird ways and from distant and terrible dimensions, echoes could be heard: *"Cthulhu fhtagn! Cthulhu fhtagn! Iä! Shub-Niggurath! The Goat with a Thousand Young!"*

Not only had the flamingos grown a lot larger, but the radioactive sludge that had been formed in the water seemed to combine with the abysmal incantations, causing some of the flamingos to mutate and start forming tentacles. When using resonance amplifiers to increase the power of the echo from beyond, the tentaclification seemed to depend both on the intensity of the echoes and the level of radiation observed. It was unclear though, if administered growth hormone had any effect on the tentacles. To answer the obvious questions, a model was proposed:

$$\text{Model:} \quad Y = \beta_0 + \beta_1 a + \beta_2 r + \beta_3 h + \epsilon,$$

where $\epsilon \sim N(0, \sigma^2)$ and different measurements are assumed to be independent. The quantity a is amplifier power and r is the radioactive intensity (in suitable units). The growth hormone is a binary where $h = 1$ means that growth hormone has been administered. The variables and measurements can be found below.

y	a	r	h
59.72	10	9	0
10.70	0	9	1
21.95	4	3	0
28.65	4	8	0
48.18	8	8	1
41.42	8	2	1
12.51	2	1	1
32.28	5	6	0
33.03	4	12	1
33.67	6	4	0

The abstract unit used to describe how tentaclified the flamingos had become was denoted tentacliness. The following calculations has already been carried out.

i	$\widehat{\beta}_i$	$d(\widehat{\beta}_i)$
0	0.41	0.82
1	4.86	0.10
2	1.11	0.08
3	0.28	0.56

Analysis of variance

	Degrees of freedom	Square sum
REGR	3	2063.96
RES	6	4.35
TOT	9	2068.32

$$(X^T X)^{-1} = \begin{pmatrix} 931 & -74 & -50 & -284 \\ -74 & 13 & 0 & 19 \\ -50 & 0 & 9 & -4 \\ -284 & 19 & -4 & 428 \end{pmatrix} \cdot 10^{-3}.$$

(a) Find a prediction interval for Y, with 90% degree of confidence, when $a = 2$, $r = 5$ and $h = 0$. (2p)

(b) Test the hypothesis that the addition of growth hormone has an effect at the significance level 1%. (1p)

5. The effect on the flamingos seemed to be the at its worst at a specific place near the edge of the water, where moving a flamingo was impossible due to twisting displays of noneuclidian geometry and nauseating vortexes swirling like maelstroms of bent light. Rick and Gary managed to trap 5 flamingos in a box without seeing them clearly. The question was how many of these flamingos that had been distinctly affected by the tentaclification process. To investigate the matter, the following procedure was carried out.

   Their friend Susan — who seemed to be less affected by everything — put her hand inside the box while keeping her eyes away to avoid madness. She then grabs hold of a random flamingo in the box and feels for tentacles. Then she immediately releases the flamingo (still inside the box). This process is repeated n times, each repetition independent of the previous ones. Let $X_i = 0$ if the flamingo in try i was unaffected and let $X_i = 1$ if it was tentaclified. Furthermore, let θ be the total number of affected flamingos in the box.

   (a) When carrying out the first 7 tries, the result was $x = 1, 0, 1, 0, 1, 1, 0$. Find the maximum likelihood estimation $\widehat{\theta}$ of θ in this instance. (2p)

   (b) Choose one of the following questions (and answer it). (1p)

        i. Prove that $\widehat{\Theta}$ is unbiased.

        ii. Prove that $\widehat{\Theta}$ is biased.

        iii. Argue for why the previous questions are difficult to answer.

6. Since it apparently was the day of the tentacle (DoTT), Rick and Gary wondered if the tentacle lengths on the different affected flamingos was independent. Rick pointed out that the lengths should be normally distributed and Gary thought that they might find the correlation between different lengths. Looking at some theorems, they find that uncorrelated normally distributed variables are independent!

   Let $X_1, X_2, \ldots, X_n \sim N(0, 1)$ be independent and let $\boldsymbol{A} \in \mathbf{R}^{n \times n}$ be invertible. Moreover, let $\boldsymbol{\mu} \in \mathbf{R}^n$. Show that the components in $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{\mu}$, where $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, are independent if and only if the covariance matrix of $\boldsymbol{Y}$ is a diagonal matrix. (2p)

Ia! Ia! Cthulhu ftagnh!