TAMS32 STOKASTISKA PROCESSER
Komplettering 1

Torkel Erhardsson

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- MARTINGALES AND OPTIONAL STOPPING

1 Definition of a martingale

Definition 1.1 *Let* $\{Z_n\}_{n=0}^{\infty}$ *be a random sequence. A random sequence* $\{X_n\}_{n=0}^{\infty}$ *is called a* **martingale** *with respect to* $\{Z_n\}_{n=0}^{\infty}$ *if: (i)* $E(|X_n|) < \infty$ *for each* $n \geq 0$*; (ii)* $X_n = g_n(Z_0, Z_1, \ldots, Z_n)$ *for some function* g_n*, for each* $n \geq 0$*; and*

$$(iii) \qquad E(X_n|Z_0, Z_1, \ldots, Z_{n-1}) = X_{n-1} \qquad \forall n \geq 1. \tag{1.1}$$

Condition (iii) is called the **martingale property***.*

Another way to express condition (ii) is to say that the value of X_n is determined by the value of $(Z_0, \ldots, Z_n)$, for each $n \geq 1$.

Example 1.1 Let $\{Z_n\}_{n=0}^{\infty}$ be independent random variables with $E(|Z_n|) < \infty$ and $E(Z_n) = 0$. Define $X_0 = 0$ and

$$X_n = \sum_{i=1}^{n} Z_i \qquad \forall n \geq 1.$$

Then $\{X_n\}_{n=1}^{\infty}$ is a martingale with respect to $\{Z_n\}_{n=0}^{\infty}$. To prove this, we check the conditions in Definition 1.1. Condition (i) holds, since

$$E(|X_n|) = E(|\sum_{i=1}^{n} Z_i|) \leq E(\sum_{i=1}^{n} |Z_i|) = \sum_{i=1}^{n} E(|Z_i|) < \infty.$$

Condition (ii) obviously also holds (why?). Finally, condition (iii) holds, since

$$E(X_n|Z_0, Z_1, \ldots, Z_{n-1}) = E(Z_n + X_{n-1}|Z_0, Z_1, \ldots, Z_{n-1}) =$$
$$= E(Z_n|Z_0, Z_1, \ldots, Z_{n-1}) + E(X_{n-1}|Z_0, Z_1, \ldots, Z_{n-1}) =$$
$$= E(Z_n) + X_{n-1} = 0 + X_{n-1} = X_{n-1}.$$

Next, we will state and prove some important theorems about martingales. The following lemma, called the **generalized law of iterated expectations**, will be useful.

Lemma 1.2 *Let* X *be a random variable such that* $E[|X|] < \infty$*. Let* Y *and* W *be random variables such that* $Y = g(W)$ *for some function* g*. Then,*

$$E(X|Y) = E(E(X|W)|Y).$$

Proof: We consider only the case when X and W (and therefore also Y) are discrete. Fix y, and let $A_y = \{w; g(w) = y\}$. What is the conditional pmf of W given $Y = y$? Clearly, for each $w \in A_y$,

$$P(W = w|Y = y) = \frac{P(\{W = w\} \cap \{Y = y\})}{P(Y = y)} = \frac{P(W = w)}{P(Y = y)},$$

while $P(W = w|Y = y) = 0$ for each $w \notin A_y$. This gives:

$$E(X|Y = y) = \sum_{x \in \mathbb{R}} xP(X = x|Y = y)$$

$$= \sum_{x \in \mathbb{R}} x \frac{P(\{X = x\} \cap \{Y = y\})}{P(Y = y)} = \sum_{x \in \mathbb{R}} x \sum_{w \in A_y} \frac{P(\{X = x\} \cap \{W = w\})}{P(Y = y)}$$

$$= \sum_{x \in \mathbb{R}} x \sum_{w \in A_y} \frac{P(X = x|W = w)P(W = w)}{P(Y = y)}$$

$$= \sum_{w \in A_y} \sum_{x \in \mathbb{R}} xP(X = x|W = w) \frac{P(W = w)}{P(Y = y)}$$

$$= \sum_{w \in A_y} E(X|W = w) \frac{P(W = w)}{P(Y = y)} = E(E(X|W)|Y = y).$$

∎

Theorem 1.3 *Let the random sequence* $\{X_n\}_{n=0}^{\infty}$ *be a martingale with respect to* $\{Z_n\}_{n=0}^{\infty}$. *Then,*

$$\begin{aligned} &(i) \quad E(X_{n+m}|Z_0, Z_1, \ldots, Z_n) = X_n \qquad && \forall n \geq 0, m \geq 0; \\ &(ii) \quad E(X_n) = E(X_0) && \forall n \geq 0. \end{aligned}$$

Proof: For (i), we use Lemma 1.2 repeatedly: first with $X = X_{n+m}$, $Y = (Z_0, Z_1, \ldots, Z_n)$ and $W = (Z_0, Z_1, \ldots, Z_{n+m-1})$, then with $X = X_{n+m-1}$, $Y = (Z_0, Z_1, \ldots, Z_n)$ and $W = (Z_0, Z_1, \ldots, Z_{n+m-2})$, and so on. Using the martingale property, we get:

$$E(X_{n+m}|Z_0, Z_1, \ldots, Z_n) = E(E(X_{n+m}|Z_0, Z_1, \ldots, Z_{n+m-1})|Z_0, Z_1, \ldots, Z_n)$$

$$= E(X_{n+m-1}|Z_0, Z_1, \ldots, Z_n) = E(E(X_{n+m-1}|Z_0, Z_1, \ldots, Z_{n+m-2})|Z_0, Z_1, \ldots, Z_n)$$

$$= E(X_{n+m-2}|Z_0, Z_1, \ldots, Z_n) = \ldots = E(X_n|Z_0, Z_1, \ldots, Z_n) = X_n.$$

(ii) follows from (i) and the law of iterated expectations by taking the expectation on both sides. ∎

Theorem 1.4 *Let the random sequence* $\{X_n\}_{n=0}^{\infty}$ *be a martingale with respect to* $\{Z_n\}_{n=0}^{\infty}$*. Then,* $\{X_n\}_{n=0}^{\infty}$ *is also a martingale with respect to* $\{X_n\}_{n=0}^{\infty}$ *(that is, with respect to itself).*

Proof: Since $\{X_n\}_{n=0}^{\infty}$ is a martingale with respect to $\{Z_n\}_{n=0}^{\infty}$, condition (i) in Definition 1.1 is satisfied. Also, condition (ii) is trivially satisfied, since $X_n = X_n$ for each $n \geq 1$. For (iii), we observe that, for each $n \geq 1$, we can use Lemma 1.2, with $X = X_n$, $Y = (X_0, X_1, \ldots, X_{n-1})$, and $W = (Z_0, Z_1, \ldots, Z_{n-1})$. This gives:

$$E(X_n|X_0, X_1, \ldots, X_{n-1}) = E(E(X_n|Z_0, Z_1, \ldots, Z_{n-1})|X_0, X_1, \ldots, X_{n-1})$$

$$= E(X_{n-1}|X_0, X_1, \ldots, X_{n-1}) = X_{n-1}.$$

■

A random sequence $\{X_n\}_{n=0}^{\infty}$ which is a martingale with respect to $\{X_n\}_{n=0}^{\infty}$ (that is, with respect to itself) will be called a **martingale**, for short.

We finally remark that a random sequence $\{X_n\}_{n=0}^{\infty}$ is called a **submartingale** with respect to $\{Z_n\}_{n=0}^{\infty}$ if it satisfies (i) and (ii) in Definition 1.1, and

$$(iii)' \qquad E(X_n|Z_0, Z_1, \ldots, Z_{n-1}) \geq X_{n-1} \qquad \forall n \geq 1. \tag{1.2}$$

$\{X_n\}_{n=0}^{\infty}$ is called a **supermartingale** with respect to $\{Z_n\}_{n=0}^{\infty}$ if it satisfies (i), (ii) and

$$(iii)'' \qquad E(X_n|Z_0, Z_1, \ldots, Z_{n-1}) \leq X_{n-1} \qquad \forall n \geq 1. \tag{1.3}$$

Clearly, a random sequence is a martingale with respect to $\{Z_n\}_{n=0}^{\infty}$ if and only if it is both a submartingale and a supermartingale with respect to $\{Z_n\}_{n=0}^{\infty}$.

2 Doob's inequality

Theorem 2.1 *Let the random sequence* $\{X_n\}_{n=0}^{\infty}$ *be a martingale. Then, for every* $\epsilon > 0$ *and for any* $n \geq 1$*,*

$$P(\max_{0 \leq k \leq n} |X_k| \geq \epsilon) \leq \frac{E(X_n^2)}{\epsilon^2}.$$

Proof: For $0 \leq j \leq n$ we have

$$\left\{ \max_{0 \leq k \leq n} |X_k| \geq \epsilon \right\} = \bigcup_{j=0}^{n} A_j,$$

where $A_j = \{|X_0| < \epsilon, |X_1| < \epsilon, \ldots, |X_{j-1}| < \epsilon, |X_j| \geq \epsilon\}$. We define the so-called indicator random variables for the events A_j, as follows:

$$I_{A_j} = \begin{cases} 1 & \text{if } A_j \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Since $0 \leq \sum_{j=0}^{n} I_{A_j} \leq 1$, we get

$$E(X_n^2) \geq E(X_n^2 \sum_{j=0}^{n} I_{A_j}) = \sum_{j=0}^{n} E(X_n^2 I_{A_j}),$$

and since $X_n^2 = (X_j + (X_n - X_j))^2$ for each $j = 0, \ldots, n$, we get

$$\begin{aligned} E(X_n^2) &\geq \sum_{j=0}^{n} E(X_j^2 I_{A_j}) + 2 \sum_{j=0}^{n} E(X_j(X_n - X_j) I_{A_j}) \\ &\quad + \sum_{j=0}^{n} E((X_n - X_j)^2 I_{A_j}) \\ &\geq \sum_{j=0}^{n} E(X_j^2 I_{A_j}) + 2 \sum_{j=0}^{n} E(X_j(X_n - X_j) I_{A_j}). \end{aligned}$$

Using the law of iterated expectations, and the fact that I_{A_j} is determined by the value of $(X_0, \ldots, X_j)$,

$$\begin{aligned} E(X_j(X_n - X_j) I_{A_j}) &= E(E(X_j I_{A_j}(X_n - X_j) | X_0, \ldots, X_j)) \\ &= E(X_j I_{A_j} E((X_n - X_j) | X_0, \ldots, X_j)), \end{aligned}$$

where

$$E((X_n - X_j)|X_0, \ldots, X_j) = E(X_n|X_0, \ldots, X_j) - X_j = X_j - X_j = 0$$

by the martingale property. Hence we get

$$E(X_n^2) \geq \sum_{j=0}^{n} E(X_j^2 I_{A_j}) \geq \epsilon^2 \sum_{j=0}^{n} E(I_{A_j})$$

since $X_j^2 \geq \epsilon^2$ when $I_{A_j} = 1$. Also $E(I_{A_j}) = P(A_j)$. Hence we have

$$\sum_{j=0}^{n} E(I_{A_j}) = P\left(\bigcup_{j=0}^{n} A_j\right),$$

since the sets A_j are disjoint. But $P\left(\bigcup_{j=0}^{n} A_j\right) = P\left(\max_{0\leq k\leq n} |X_k| \geq \epsilon\right)$. Hence we have proved

$$E(X_n^2) \geq \epsilon^2 P\left(\max_{0\leq k\leq n} |X_k| \geq \epsilon\right),$$

which is the assertion in the theorem. ∎

It should be noted that Doob's inequality is stronger than Markov's inequality, which states that, for each $\epsilon > 0$,

$$P(|X_n| \geq \epsilon) \leq \frac{E(X_n^2)}{\epsilon^2} \qquad \forall n \geq 0.$$

On the other hand, Markov's inequality holds not only for martingales but for any random sequence $\{X_n\}_{n=0}^{\infty}$.

3 Stopping times and optional stopping

Definition 3.1 *Let $\{Z_n\}_{n=0}^{\infty}$ be a random sequence. A random variable T taking values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a* **stopping time** *with respect to $\{Z_n\}_{n=0}^{\infty}$, if $I\{T = n\} = g_n(Z_0, Z_1, \ldots, Z_n)$ for some function g_n, for each $n \geq 0$. Here, $I\{T = n\}$ is the indicator random variable for the event $\{T = n\}$, defined by*

$$I\{T = n\} = \begin{cases} 1 & \text{if } \{T = n\} \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

In words, T is a stopping time with respect to $\{Z_n\}_{n=0}^{\infty}$ if, for each $n \geq 0$, it is possible to decide from the value of $(Z_0, Z_1, \ldots, Z_n)$ whether $\{T = n\}$ occurs or not.

Example 3.1 Let $\{Z_n\}_{n=0}^{\infty}$ be a random sequence, and let $a \in \mathbb{R}$. Then, the random variable $T_a = \inf\{n \geq 1; Z_n \geq a\}$ is a stopping time with respect to $\{Z_n\}_{n=0}^{\infty}$, since $\{T_a = 0\}$ never occurs, and

$$\{T_a = n\} = \{Z_1 < a, Z_2 < a, \ldots, Z_n \geq a\} \qquad \forall n \geq 1.$$

On the other hand, the random variable $S_a = \sup\{n \geq 0; Z_n \geq a\}$ is *not* a stopping time with respect to $\{Z_n\}_{n=0}^{\infty}$, since it cannot always be decided just from the value of $(Z_0, Z_1, \ldots, Z_n)$ whether $\{S_a = n\}$ occurs or not: if $Z_n \geq a$, the values of $Z_{n+1}, Z_{n+2}, \ldots$ also matter.

Theorem 3.2 *Let S and T be stopping times with respect to the random sequence $\{Z_n\}_{n=0}^{\infty}$. Then, $S+T$, $\max\{S,T\}$ and $\min\{S,T\}$ are also stopping times with respect to $\{Z_n\}_{n=0}^{\infty}$.*

Proof: We prove only the second claim. Since

$$\{\max\{S,T\} = n\} = \big(\{S = n\} \cap (\cup_{k=0}^{n}\{T = k\})\big) \cup \big(\{T = n\} \cap (\cup_{k=0}^{n}\{S = k\})\big),$$

and since S and T are stopping times, it can be decided from the value of $(Z_0, Z_1, \ldots, Z_n)$ whether $\{\max\{S,T\} = n\}$ occurs or not. ∎

Next, we will show that a martingale which is "stopped" at a stopping time T is still a martingale. To do this, we will need the following lemma.

Lemma 3.3 *Let the random sequence $\{X_n\}_{n=0}^{\infty}$ be a martingale with respect to $\{Z_n\}_{n=0}^{\infty}$. Let the random sequence $\{H_n\}_{n=1}^{\infty}$ be such that: (i) $|H_n| \leq C_n < \infty$ for each $n \geq 1$ (where C_n is a constant), and: (ii) $H_n = h_n(Z_0, \ldots, Z_{n-1})$ for some function h_n, for each $n \geq 1$. Define the random sequence $\{Y_n\}_{n=0}^{\infty}$ by $Y_0 = X_0$, and*

$$Y_n = \sum_{i=1}^{n} H_i(X_i - X_{i-1}) + X_0 \qquad \forall n \geq 1.$$

Then, $\{Y_n\}_{n=0}^{\infty}$ is a martingale with respect to $\{Z_n\}_{n=0}^{\infty}$.

Proof: Condition (i) in Definition 1.1 holds, since

$$E(|Y_n|) = E(|\sum_{i=1}^{n} H_i(X_i - X_{i-1})|) \leq \sum_{i=1}^{n} E(|H_i(X_i - X_{i-1})|)$$

$$\leq \sum_{i=1}^{n} C_i E(|X_i - X_{i-1}|) = \sum_{i=1}^{n} C_i E(|X_i| + |X_{i-1}|) < \infty.$$

Condition (ii) clearly also holds (why?), and condition (iii) holds since

$$E(Y_n | Z_0, \ldots, Z_{n-1}) = E(Y_{n-1} + H_n(X_n - X_{n-1}) | Z_0, \ldots, Z_{n-1})$$

$$= Y_{n-1} + E(H_n(X_n - X_{n-1})|Z_0, \ldots, Z_{n-1})$$
$$= Y_{n-1} + H_n(E(X_n|Z_0, \ldots, Z_{n-1}) - X_{n-1}) = Y_{n-1}.$$

■

A typical application of Lemma 3.3 is given in the following example.

Example 3.4 Let $\{Z_n\}_{n=1}^{\infty}$ be independent random variables with $E(|Z_n|) < \infty$ and $E(Z_n) = 0$. Each of these random variables is the outcome of a game. Define $X_0 = 0$ and

$$X_n = \sum_{i=1}^{n} Z_i \qquad \forall n \geq 1.$$

Then, we know from Example 1.1 that $\{X_n\}_{n=1}^{\infty}$ is a martingale. Suppose that, *immediately before* the nth game, a player can decide to join the game by betting an amount H_n, which is bounded by a constant $C < \infty$, but may depend on the outcomes of all the preceding games. The player's net gain from the nth game is then $H_n Z_n = H_n(X_n - X_{n-1})$, and the player's total net gain from the first n games is

$$Y_n = \sum_{i=1}^{n} H_i(X_i - X_{i-1}) \qquad \forall n \geq 1.$$

Lemma 3.3 says that "you can't beat a fair game": no matter how you choose your bets $\{H_n\}_{n=1}^{\infty}$, your total net gain will be a martingale with mean $E(Y_0) = E(X_0) = 0$.

Theorem 3.5 *Let the random sequence $\{X_n\}_{n=0}^{\infty}$ be a martingale with respect to $\{Z_n\}_{n=0}^{\infty}$. Let T be a stopping time with respect to $\{Z_n\}_{n=0}^{\infty}$. Define the random sequence $\{Y_n\}_{n=0}^{\infty}$ by*

$$Y_n = X_{\min\{T,n\}} \qquad \forall n \geq 0.$$

Then, $\{Y_n\}_{n=0}^{\infty}$ is a martingale with respect to $\{Z_n\}_{n=0}^{\infty}$. It is called a **stopped martingale**.

Proof: Let $H_n = I\{T \geq n\}$ for each $n \geq 1$. Then, the random sequence $\{H_n\}_{n=1}^{\infty}$ satisfies the two conditions of Lemma 3.3 (condition (*ii*) since $H_n = 1 - I\{T \leq n-1\}$). Moreover,

$$Y_n = X_{\min\{T,n\}} = \sum_{i=1}^{n} H_i(X_i - X_{i-1}) + X_0 \qquad \forall n \geq 0,$$

so the claim follows from Lemma 3.3. ∎

Theorem 3.6 (Optional stopping) *Let the random sequence $\{X_n\}_{n=0}^{\infty}$ be a martingale with respect to $\{Z_n\}_{n=0}^{\infty}$. Let T be a stopping time with respect to $\{Z_n\}_{n=0}^{\infty}$. Assume that either: (i) $T \leq a < \infty$ (where a is a constant), or: (ii) $E(X_n^2) \leq C < \infty$ for all $n \geq 0$ (where C is a constant not depending on n). Then,*

$$E(X_T) = E(X_0).$$

Proof: In the case when (i) holds,

$$E(X_T) = E(X_{\min\{T,a\}}) = E(X_{\min\{T,0\}}) = E(X_0),$$

where the second equality follows from Theorem 1.3 and the fact that a stopped martingale is a martingale. The case when (ii) holds is omitted. ∎

Example 3.7 Let $\{Z_n\}_{n=1}^{\infty}$ be independent identically distributed random variables, such that $P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2}$. Define $X_0 = 0$ and

$$X_n = \sum_{i=1}^{n} Z_i \qquad \forall n \geq 1.$$

The random sequence $\{X_n\}_{n=0}^{\infty}$ is called a **simple symmetric random walk**, and we know from Example 1.1 that it is a martingale. Let $a < 0 < b$, and let $T = \inf\{n \geq 1; X_n = a \text{ or } X_n = b\}$. T is a stopping time (why?), and $T < \infty$, since $\{Z_n\}_{n=1}^{\infty}$ will contain at least one subsequence of length $b - a$ consisting only of 1s (prove this!), which will drive $\{X_n\}_{n=1}^{\infty}$ out of the interval (a, b). Therefore, X_T can only take the values a or b. We would like to compute $P(X_T = b) = p$.

To do this, consider the stopped martingale $\left\{X_{\min\{T,n\}}\right\}_{n=0}^{\infty}$. Since this martingale can only take values in the interval $[a, b]$, condition (ii) in Theorem 3.6 is satisfied (with $C = \max\{a^2, b^2\}$). Theorem 3.6 therefore gives:

$$E(X_{\min\{T,T\}}) = E(X_T) = aP(X_T = a) + bP(X_T = b) = a(1-p) + bp$$

$$= E(X_0) = 0 \quad \Rightarrow \quad p = \frac{-a}{b-a}.$$

4 Problems

1. Let $\{X_n\}_{n=1}^{\infty}$ be independent and identically distributed random variables, with $E(X_n) = \mu$ and $V(X_n) = \sigma^2$. Define

$$W_0 = 0, W_n = \sum_{i=1}^{n} X_i$$

and

$$S_n = (W_n - n\mu)^2 - n\sigma^2.$$

Show that $\{S_n\}_{n=0}^{\infty}$ is a martingale. (That is: show that $\{S_n\}_{n=0}^{\infty}$ is a martingale with respect to some suitably chosen underlying random sequence, for example $\{X_n\}_{n=1}^{\infty}$. It then follows by Theorem 1.4 that $\{S_n\}_{n=0}^{\infty}$ is also a martingale with respect to itself.)

2. Let X be a random variable such that $E(|X|) < \infty$, and let $\{Z_n\}_{n=0}^{\infty}$ be a random sequence. Define the random sequence $\{X_n\}_{n=0}^{\infty}$ by

$$X_n = E(X|Z_0, Z_1, \ldots, Z_n) \qquad \forall n \geq 0.$$

Show that $\{X_n\}_{n=0}^{\infty}$ is a martingale. (You need not show that the condition $E(|X_n|) < \infty$ holds.)

3. Let $\{B(t); t \geq 0\}$ be a Brownian motion. Let $X_n = B(t_n)$ for $0 = t_0 < t_1 < \ldots < t_n < \ldots$. Show that $\{X_n\}_{n=0}^{\infty}$ is a martingale.

4. Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of random variables. In many statistical applications, it is assumed that $\{X_n\}_{n=0}^{\infty}$ are independent and identically distributed, and such that their pdf:s are either ψ or ϕ. This means that the joint pdf of $(X_0, X_1, \ldots, X_n)$ is either

$$\psi_{X_0,X_1,\ldots,X_n}(x_0, x_1, \ldots, x_n) = \psi(x_0)\,\psi(x_1)\cdots\psi(x_n)$$

or

$$\phi_{X_0,X_1,\ldots,X_n}(x_0, x_1, \ldots, x_n) = \phi(x_0)\,\phi(x_1)\cdots\phi(x_n).$$

The *likelihood ratio* L_n is defined as

$$L_n = \frac{\psi(x_0)\,\psi(x_1)\cdots\psi(x_n)}{\phi(x_0)\,\phi(x_1)\cdots\phi(x_n)}.$$

where we assume that $\phi(x) > 0$ for all x. Show that L_n is a martingale with respect to $\{X_n\}_{n=0}^{\infty}$, if $\{X_n\}_{n=0}^{\infty}$ are independent and identically distributed random variables with pdf ϕ.

Remark: Under the condition just mentioned, it can be proven that $L_n \to 0$, as $n \to 0$ (you are not required to prove this). What does this mean in terms of choosing between ψ and ϕ as models for $\{X_n\}_{n=0}^{\infty}$, using the likelihood ratio as a criterion ?

5. Prove the two remaining claims in Theorem 3.2.

6. Just as in Example 3.7, let $\{Z_n\}_{n=1}^{\infty}$ be independent identically distributed random variables, such that $P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2}$. Define $X_0 = 0$ and

$$X_n = \sum_{i=1}^{n} Z_i \qquad \forall n \geq 1.$$

The random sequence $\{X_n\}_{n=0}^{\infty}$ is a martingale with respect to $\{Z_n\}_{n=1}^{\infty}$. Let $a < 0 < b$, and let $T = \inf\{n \geq 0; X_n = a \text{ or } X_n = b\}$. It was pointed out in Example 3.7 that T is a stopping time, and that $P(T < \infty) = 1$.

(a) Determine the constant c so that $\{Y_n\}_{n=0}^{\infty}$, defined by

$$Y_n = (b - X_n)(X_n - a) + cn \qquad \forall n \geq 0,$$

is a martingale with respect to $\{Z_n\}_{n=1}^{\infty}$.

(b) Use this martingale to show that $E(T) = -ab$. (*Hint:* $T_n = \min(T, n)$ is also a stopping time. Furthermore, you may use that $E(T_n) \to E(T)$, $E(X_{T_n}) \to E(X_T)$, and $E(X_{T_n}^2) \to E(X_T^2)$, as $n \to \infty$. No proofs of these results are required.)

7. Let $Z_0 = 0$, and let $\{Z_n\}_{n=1}^{\infty}$ be independent and identically distributed random variables, such that

$$P(Z_n = 2) = P(Z_n = 0) = \frac{1}{2} \qquad \forall n \geq 1.$$

Define the random sequence $\{X_n\}_{n=0}^{\infty}$ by $X_0 = 1$ and

$$X_n = \prod_{i=1}^{n} Z_i \qquad \forall n \geq 1.$$

(a) Verify that $\{X_n\}_{n=0}^{\infty}$ is a martingale.

(b) Define the stopping time $T = \min\{n \geq 1; X_n = 0\}$. Can you prove, using the optional stopping theorem, that $E(X_T) = E(X_0)$? If not, what is the problem?

8. Let $\{X_n\}_{n=0}^{\infty}$ be a **nonnegative** martingale with $E(X_0) = 1$, and let $a > 0$. Prove that, for any fixed $n \geq 0$,

$$P(X_k \geq a \text{ for some } 0 \leq k \leq n) \leq \frac{1}{a}.$$

Hint: Use Markov's inequality and the optional stopping theorem with an appropriately chosen bounded stopping time.