TAMS32 STOKASTISKA PROCESSER
Komplettering 2

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- MEAN SQUARE CONVERGENCE.
- MEAN SQUARE CONTINUITY.
- MEAN SQUARE INTEGRALS.

1 Definition of convergence in mean square and in probability

Definition 1.1 *A random sequence* $\{X_n\}_{n=1}^{\infty}$ *such that* $E(X_n^2) < \infty$ *for each* $n \geq 1$ *is said to* **converge in mean square** *to a random variable* X*, if* $E(X^2) < \infty$ *and*

$$E((X_n - X)^2) \to 0 \qquad \text{as } n \to \infty.$$

In Swedish this is called *konvergens i (kvadratiskt) medel*. We write also

$$X = \underset{n\to\infty}{\text{l.i.m.}}\ X_n.$$

Note that this definition does not say anyhing about the (possible) convergence of the *sample paths* of $\{X_n\}_{n=1}^{\infty}$ as $n \to \infty$.

Definition 1.2 *A random sequence* $\{X_n\}_{n=1}^{\infty}$ *is said to* **converge in probability** *to a random variable* X*, if*

$$P(|X_n - X| > \epsilon) \to 0 \qquad \text{as } n \to \infty \ \ \forall \epsilon > 0.$$

Theorem 1.1 *If a random sequence* $\{X_n\}_{n=1}^{\infty}$ *converges in mean square to a random variable* X*, then it also converges in probability to* X*.*

Proof: Chebyshev's inequality gives:

$$P(|X_n - X| > \epsilon) \leq \frac{E((X_n - X)^2)}{\epsilon^2} \qquad \forall \epsilon > 0,$$

from which the result immediately follows. ∎

2 Laws of large numbers

In the so-called *mean square law of large numbers*, we have convergence in mean square to a degenerate random variable, i.e., a constant:

Theorem 2.1 *Let the random variables* $\{X_n\}_{n=1}^{\infty}$ *be uncorrelated (meaning that* $C(X_i, X_j) = 0$ *for all* $i \neq j$*), and such that* $E(X_n) = \mu < \infty$ *for each* $n \geq 1$ *and* $V(X_n) = \sigma^2 < \infty$ *for each* $n \geq 1$*. Then*

$$\mu = \underset{n\to\infty}{\text{l.i.m.}}\ \frac{1}{n} \sum_{j=1}^{n} X_j.$$

Proof: Let us set $S_n = \frac{1}{n}\sum_{j=1}^{n} X_j$. We have $E(S_n) = \mu$ and $Var(S_n) = \frac{1}{n}\sigma^2$, since the variables are uncorrelated. For the claimed mean square convergence we need to consider

$$E((S_n - \mu)^2) = E((S_n - E(S_n))^2) = Var(S_n) = \frac{1}{n}\sigma^2$$

so that

$$E((S_n - \mu)^2) = \frac{1}{n}\sigma^2 \to 0$$

as $n \to \infty$, as was claimed. ■

Since convergence in mean square implies convergence in probability, we also have the *weak law of large numbers*:

Theorem 2.2 *Let the random variables $\{X_n\}_{n=1}^{\infty}$ be uncorrelated, and such that $E(X_n) = \mu < \infty$ for each $n \geq 1$ and $Var(X_n) = \sigma^2 < \infty$ for each $n \geq 1$. Then*

$$\frac{1}{n}\sum_{j=1}^{n} X_j \to \mu$$

in probability, as $n \to \infty$.

3 Some Useful Inequalities for Random Variables

Lemma 3.1 *For any random variable X,*

$$|E(X)| \leq E(|X|). \tag{3.1}$$

Proof: If X is a continuous random variable with probability density function $f_X(x)$, then by a result from the basic analysis course:

$$|E(X)| = |\int_{-\infty}^{\infty} x f_X(x)dx| \leq \int_{-\infty}^{\infty} |x| f_X(x)dx = E(|X|).$$

The case when X is a discrete random variable is shown analogously. ■

Lemma 3.2

$$E(|XY|) \leq \sqrt{E(X^2)E(Y^2)}. \tag{3.2}$$

$$\sqrt{E((X+Y)^2)} \leq \sqrt{E(X^2)} + \sqrt{E(Y^2)}. \tag{3.3}$$

(3.2) is known as the *Cauchy inequality for random variables*, and (3.3) is known as the *triangle inequality for random variables.*

Proof: (3.2) follows from the inequality

$$0 \le E\big(\big(\frac{|X|}{\sqrt{E(X^2)}} - \frac{|Y|}{\sqrt{E(Y^2)}}\big)^2\big) = E\big(\frac{X^2}{E(X^2)} + \frac{Y^2}{E(Y^2)} - \frac{2|XY|}{\sqrt{E(X^2)E(Y^2)}}\big)$$

$$= 1 + 1 - \frac{2E(|XY|)}{\sqrt{E(X^2)E(Y^2)}} = 2\big(1 - \frac{E(|XY|)}{\sqrt{E(X^2)E(Y^2)}}\big),$$

by rearranging the terms. (3.3) follows from

$$E((X+Y)^2) \le E((|X|+|Y|)|X+Y|) = E(|X||X+Y|) + E(|Y||X+Y|)$$

$$\le \sqrt{E(X^2)E((X+Y)^2)} + \sqrt{E(Y^2)E((X+Y)^2)}$$

$$= \sqrt{E((X+Y)^2)}\big(\sqrt{E(X^2)} + \sqrt{E(Y^2)}\big),$$

after we divide both sides with $\sqrt{E((X+Y)^2)}$. Here, the Cauchy inequality was used in the third step. ■

4 Properties of mean square convergence

Theorem 4.1 *Let the random sequences $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be such that $E(X_n^2) < \infty$ and $E(Y_n^2) < \infty$ for each $n = 1, 2, \ldots$, and such that*

$$X = \underset{n\to\infty}{\text{l.i.m.}}\, X_n, \quad Y = \underset{n\to\infty}{\text{l.i.m.}}\, Y_n.$$

Then,

(a) $E(X_n) \to E(X)$ as $n \to \infty$;

(b) $E(X_n^2) \to E(X^2)$ as $n \to \infty$;

(c) $E(X_nY_m) \to E(XY)$ as $\min(n,m) \to \infty$;

(d) If $E(Z^2) < \infty$, then $E(X_nZ) \to E(XZ)$ as $n \to \infty$.

Proof: The reader is asked to prove (a) and (b) in Problem 1, Section 9. Moreover, (d) follows directly from (c) by choosing $Y_m = Z$ for $m = 1, 2, \ldots$. In order to prove (c), we first observe that

$$|E(X_n Y_m)| \leq E(|X_n Y_m|) \leq \sqrt{E(X_n^2) E(Y_m^2)} < \infty$$

by the Cauchy inequality and Lemma 3.1. Similarly, $|E(XY)| < \infty$. Next,

$$|E(X_n Y_m) - E(XY)| = |E(X_n Y_m - XY)|$$

$$= |E((X_n - X)Y_m + (Y_m - Y)X)| \leq E(|(X_n - X)Y_m + (Y_m - Y)|X)$$
$$\leq E(|(X_n - X)Y_m|) + E(|(Y_m - Y)|X),$$

where in the last step we used the *usual* triangle inequality for real numbers. Using the Cauchy inequality again, we get:

$$E(|(X_n - X)Y_m|) \leq \sqrt{E((X_n - X)^2) E(Y_m^2)}$$

and

$$E(|(Y_m - Y)X|) \leq \sqrt{E((Y_m - Y)^2) E(X^2)}.$$

By assumption, $E((X_n - X)^2) \to 0$ as $n \to \infty$, and $E((Y_m - Y)^2) \to 0$ as $m \to \infty$. Since the square root is a continuous function, it follows that $\sqrt{E((X_n - X)^2)} \to 0$ as $n \to \infty$, and $\sqrt{E((Y_m - Y)^2)} \to 0$ as $m \to \infty$. Finally, $E(Y_m^2) \to E(Y^2)$ by part (b), so the sequence $\{E(Y_m^2); m = 1, 2, \ldots\}$ is bounded. Hence, (c) is proved. ■

We shall often need *Cauchy's criterion for mean square convergence*, which is the next theorem.

Theorem 4.2 (Cauchy's criterion) *Let the random sequence* $\{X_n\}_{n=1}^{\infty}$ *be such that* $E(X_n^2) < \infty$ *for each* $n = 1, 2, \ldots$. *It then holds that*

$$E((X_n - X_m)^2) \to 0 \quad \text{as } \min(m, n) \to \infty \tag{4.4}$$

if and only if there exists a random variable X *such that*

$$X = \underset{n \to \infty}{\text{l.i.m.}}\ X_n.$$

Proof: Proof of $\Longleftarrow$:

$$\sqrt{E((X_n - X_m)^2)} = \sqrt{E((X_n - X + X - X_m)^2)}$$

$$\leq \sqrt{E((X_n - X)^2)} + \sqrt{E((X - X_m)^2)},$$

where the triangle inequality for random variables was used in the last step. By assumption, both terms on the right hand side go to 0 as $\min(n, m) \to \infty$.

Proof of $\Longrightarrow$: Omitted. For those who have taken a course in functional analysis, we remark that what needs to be proven is that the space of random variables (defined on the same sample space Ω) such that $E(X^2) < \infty$, with the norm $||X|| = \sqrt{E(X^2)}$, is a complete normed linear space. ∎

The following is a sometimes useful alternative to Cauchy's criterion:

Theorem 4.3 (Loève's criterion) *Let the random sequence $\{X_n\}_{n=1}^{\infty}$ be such that $E(X_n^2) < \infty$ for each $n = 1, 2, \ldots$. It then holds that*

$$E((X_n - X_m)^2) \to 0 \quad \text{as } \min(m, n) \to \infty \tag{4.5}$$

if and only if there exists a finite constant C such that

$$E(X_n X_m) \to C \quad \text{as } \min(m, n) \to \infty. \tag{4.6}$$

Proof: Proof of $\Longleftarrow$: We assume that $E(X_n X_m) \to C$ as $\min(m, n) \to \infty$. Then,

$$\begin{aligned} E((X_n - X_m)^2) &= E(X_n X_n + X_m X_m - 2X_n X_m) \\ &\to C + C - 2C = 0 \quad \text{as } \min(m, n) \to \infty. \end{aligned}$$

Proof of $\Longrightarrow$: We assume that $E((X_n - X_m)^2) \to 0$ as $\min(m, n) \to \infty$. Then $X = \underset{n \to \infty}{\text{l.i.m.}}\ X_n$ exists, according to Cauchy's criterion. Using Theorem 4.1(c) and choosing $Y_m = X_m$ for $m = 1, 2, \ldots$, we get:

$$E(X_n X_m) \to E(X^2) \quad \text{as } \min(m, n) \to \infty,$$

so (4.6) holds with $C = E(X^2)$. ∎

5 Applications

Theorem 5.1 *Let the random variables $\{X_n\}_{n=0}^{\infty}$ be uncorrelated (meaning that $C(X_i, X_j) = 0$ for all $i \neq j$), and such that $E(X_n) = \mu < \infty$ for each*

$n \geq 0$ and $V(X_n) = \sigma^2 < \infty$ for each $n \geq 0$. Then, the random sequence $\{\sum_{i=0}^{n} a_i X_i\}_{n=0}^{\infty}$ converges in mean square as $n \to \infty$ to a random variable

$$\sum_{i=0}^{\infty} a_i X_i = \operatorname*{l.i.m.}_{n\to\infty} \sum_{i=0}^{n} a_i X_i$$

if and only if $\sum_{i=0}^{\infty} a_i^2 < \infty$ and $\sum_{i=0}^{\infty} a_i$ converges, where the second condition is not needed if $\mu = 0$.

Proof: Let $Y_n = \sum_{i=0}^{n} a_i X_i$ for each $n = 1, 2, \ldots$, and assume that $n < m$. Then,

$$E((Y_n - Y_m)^2) = E((\sum_{i=n+1}^{m} a_i X_i)^2) = \sigma^2 \sum_{i=n+1}^{m} a_i^2 + \mu^2 \left(\sum_{i=n+1}^{m} a_i\right)^2,$$

since $E(Z^2) = V(Z) + E(Z)^2$ for any random variable, and the random variables $\{X_n\}_{n=0}^{\infty}$ are uncorrelated. Hence we see that $E((Y_n - Y_m)^2)$ converges to 0 as $\min\{n, m\} \to \infty$ if and only if both $\sum_{i=n+1}^{m} a_i^2$ and $\sum_{i=n+1}^{m} a_i$ converge to 0 as $\min\{n, m\} \to \infty$ (where the second condition clearly is needed only if $\mu \neq 0$). By the *Cauchy criterion for sequences of real numbers*, this is equivalent to $\sum_{i=0}^{\infty} a_i^2 < \infty$ and $\sum_{i=0}^{\infty} a_i$ converges. ∎

Theorem 5.2 *Let the random sequence $\{X_n\}_{n=0}^{\infty}$ be a martingale and assume that*

$$E(X_n^2) \leq C < \infty \qquad \forall n \geq 0 \tag{5.7}$$

for some constant C. Then, $\{X_n\}_{n=0}^{\infty}$ converges in mean square to a random variable X as $n \to \infty$.

Proof: We use the Cauchy criterion. Assume that $n < m$. Then,

$$E((X_n - X_m)^2) = E(X_n^2) + E(X_m^2) - 2E(X_m X_n),$$

where

$$E(X_m X_n) = E(E(X_m X_n | X_0, \ldots, X_n)) = E(X_n E(X_m | X_0, \ldots, X_n)) = E(X_n^2),$$

by the martingale property. Hence,

$$E((X_n - X_m)^2) = E(X_m^2) - E(X_n^2) \geq 0.$$

Hence, the sequence of numbers $\{E(X_n^2); n = 0, 1, \ldots\}$ is nondecreasing. Since, by assumption, it is also bounded above by $C < \infty$, it must be convergent. This in turn implies that

$$E((X_n - X_m)^2) = E(X_m^2) - E(X_n^2) \to 0$$

as $\min\{m, n\} \to \infty$. ∎

6 Mean square continuity

Definition 6.1 *Let $\{X(t); t \geq 0\}$ be a stochastic process in continuous time. The process is said to be* mean square continuous *if*

$$E((X(t+\tau) - X(t))^2) \to 0$$

as $\tau \to 0$, for every $t \geq 0$.

Theorem 6.1 *Let $\{X(t); t \geq 0\}$ be a wide sense stationary stochastic process in continuous time. Then, the process is mean square continuous if and only if the autocorrelation function $R_X(\tau)$ is continuous at $\tau = 0$, or (equivalently) that the autocovariance function $C_X(\tau)$ is continuous at $\tau = 0$.*

Proof: For any stochastic process $\{X(t); t \geq 0\}$, we can write:

$$\begin{aligned} E((X(t+\tau) - X(t))^2) &= E(X(t+\tau)X(t+\tau)) - E(X(t+\tau)X(t)) \\ &\quad - E(X(t)X(t+\tau)) + E(X(t)X(t)) \\ &= R_X(t+\tau, 0) - R_X(t, \tau) - R_X(t, \tau) + R_X(t, 0). \end{aligned}$$

Hence, if $\{X(t); t \geq 0\}$ is wide sense stationary, then

$$E((X(t+\tau) - X(t))^2) = 2R_X(0) - 2R_X(\tau) = 2C_X(0) - 2C_X(\tau).$$

∎

7 Mean square integral

Definition 7.1 *Let $\{X(t); t \geq 0\}$ be a stochastic process in continuous time. Choose a sequence $\{\underline{t}^{(n)} = (t_0^{(n)}, t_1^{(n)}, \ldots, t_n^{(n)}); n = 1, 2, \ldots\}$ such that $a = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = b$ for each $n = 1, 2, \ldots$, and such that $\max_{i=1,\ldots,n} |t_i^{(n)} - t_{i-1}^{(n)}| \to 0$ as $n \to \infty$. Choose also a sequence $\{\underline{\xi}^{(n)} = (\xi_1^{(n)}, \ldots, \xi_n^{(n)}); n = 1, 2, \ldots\}$ such that $t_{i-1}^{(n)} \leq \xi_i^{(n)} \leq t_i^{(n)}$ for $i = 1, \ldots, n$. The* mean square integral *$\int_a^b X(t)dt$ is defined as the mean square limit*

$$\int_a^b X(t)dt = \underset{n\to\infty}{\text{l.i.m.}} \sum_{i=1}^{n} X(\xi_i^{(n)})(t_i^{(n)} - t_{i-1}^{(n)}), \tag{7.8}$$

whenever the limit exists and is independent of the choice of $\{\underline{t}^{(n)}; n = 1, 2, \ldots\}$ and $\{\underline{\xi}^{(n)}; n = 1, 2, \ldots\}$

Theorem 7.1 *The mean square integral $\int_a^b X(t)dt$ exists if and only if the double integral*

$$\int_a^b \int_a^b E(X(t)X(u))dtdu$$

exists as a Riemann integral. In this case, it also holds that

$$E(\int_a^b X(t)dt) = \int_a^b E(X(t))dt \tag{7.9}$$

and

$$E((\int_a^b X(t)dt)^2) = \int_a^b \int_a^b E(X(t)X(u))dtdu. \tag{7.10}$$

Proof: Proof of $\Longrightarrow$: Let $Y_n = \sum_{i=1}^n X(\xi_i^{(n)})(t_i^{(n)} - t_{i-1}^{(n)})$, where $\{\underline{t}^{(n)}; n = 1, 2, \ldots\}$ and $\{\underline{\xi}^{(n)}; n = 1, 2, \ldots\}$ are sequences with the properties mentioned in Definition 7.1. Let also $Z_n = \sum_{i=1}^n X(\eta_i^{(n)})(u_i^{(n)} - u_{i-1}^{(n)})$, where $\{\underline{u}^{(n)}; n = 1, 2, \ldots\}$ and $\{\underline{\eta}^{(n)}; n = 1, 2, \ldots\}$ are two other sequences with the same properties as $\{\underline{t}^{(n)}; n = 1, 2, \ldots\}$ and $\{\underline{\xi}^{(n)}; n = 1, 2, \ldots\}$. We have:

$$E(Y_n Z_m) = \sum_{i=1}^{n} \sum_{j=1}^{m} E(X(\xi_i^{(n)})X(\eta_j^{(n)}))(t_i^{(n)} - t_{i-1}^{(n)})(u_j^{(m)} - u_{j-1}^{(m)}), \tag{7.11}$$

where the right hand side is a Riemann sum. By Theorem 4.1(c),

$$E(Y_n Z_m) \to E((\int_a^b X(t)dt)^2) \qquad \text{as } \min(m, n) \to \infty,$$

where the limit does not depend on the choice of the sequences $\{\underline{t}^{(n)}; n = 1, 2, \ldots\}$, $\{\underline{\xi}^{(n)}; n = 1, 2, \ldots\}$, $\{\underline{u}^{(n)}; n = 1, 2, \ldots\}$ and $\{\underline{\eta}^{(n)}; n = 1, 2, \ldots\}$. By the definition of the Riemann integral, therefore, the double integral

$$\int_a^b \int_a^b E(X(t)X(u))dtdu$$

exists as a Riemann integral, and

$$E((\int_a^b X(t)dt)^2) = \int_a^b \int_a^b E(X(t)X(u))dtdu.$$

By Theorem 4.1(a), we also get

$$E(Y_n) = \sum_{i=1}^{n} E(X(\xi_i^{(n)}))(t_i^{(n)} - t_{i-1}^{(n)}) \to E(\int_a^b X(t)dt) \quad \text{as } n \to \infty,$$

where the limit does not depend on the choice of the sequences $\{\underline{t}^{(n)}; n = 1, 2, \ldots\}$ and $\{\underline{\xi}^{(n)}; n = 1, 2, \ldots\}$. Therefore, $\int_a^b E(X(t))dt$ exists as a Riemann integral, and

$$E(\int_a^b X(t)dt) = \int_a^b E(X(t))dt.$$

Proof of $\Longleftarrow$: We define Y_n and Z_n as before. It then holds that

$$E(Y_n Y_m) = \sum_{i=1}^{n} \sum_{j=1}^{m} E(X(\xi_i^{(n)})X(\xi_j^{(n)}))(t_i^{(n)} - t_{i-1}^{(n)})(t_j^{(m)} - t_{j-1}^{(m)}).$$

The existence of the Riemann integral implies that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} E(X(\xi_i^{(n)})X(\xi_j^{(n)}))(t_i^{(n)} - t_{i-1}^{(n)})(t_j^{(m)} - t_{j-1}^{(m)})$$

$$\to \int_a^b \int_a^b E(X(t)X(u))dtdu \qquad \text{as } \min(m, n) \to \infty.$$

By Loève's criterion, this implies that $Y = \operatorname*{l.i.m.}_{n\to\infty} Y_n$ exists, and by Theorem 4.1(b),

$$E(Y^2) = \int_a^b \int_a^b E(X(t)X(u))dtdu,$$

where the right hand side does not depend on the choice of the sequences $\{\underline{t}^{(n)}; n = 1, 2, \ldots\}$ and $\{\underline{\xi}^{(n)}; n = 1, 2, \ldots\}$. To show that the random variable Y does not in any way depend on the choice of the sequences $\{\underline{t}^{(n)}; n = 1, 2, \ldots\}$ and $\{\underline{\xi}^{(n)}; n = 1, 2, \ldots\}$, let $Z = \underset{n\to\infty}{\text{l.i.m.}}\ Z_n$, and compute

$$E((Y - Z)^2) = E(Y^2) + E(Z^2) - 2E(YZ).$$

We have already seen that $E(Z^2) = E(Y^2)$. From equation (7.11), Theorem 4.1(d), and since the existence of the Riemann integral implies that

$$E(Y_n Z_m) = \sum_{i=1}^{n} \sum_{j=1}^{m} E(X(\xi_i^{(n)})X(\eta_j^{(n)}))(t_i^{(n)} - t_{i-1}^{(n)})(u_j^{(m)} - u_{j-1}^{(m)})$$

$$\to \int_a^b \int_a^b E(X(t)X(u))dtdu \qquad \text{as } \min(m, n) \to \infty,$$

we get that $E(YZ) = E(Y^2)$. Therefore, $E((Y - Z)^2) = 0$, which implies that $P(Y = Z) = 1$. ∎

It turns out that mean square integrals obey many of the same rules as ordinary Riemann integrals.

Theorem 7.2 *(a)*

$$\int_a^b (\alpha X(t) + \beta Y(t))\, dt = \alpha \int_a^b X(t)dt + \beta \int_a^b Y(t)dt$$

(b)

$$\int_a^b X(t)dt + \int_b^c X(t)dt = \int_a^c X(t)dt$$

Proof: Omitted.

8 Problems

1. Let the random sequence $\{X_n\}_{n=1}^{\infty}$ be such that $E(X_n^2) < \infty$ for each $n = 1, 2, \ldots$, and assume that

$$X = \underset{n\to\infty}{\text{l.i.m.}}\ X_n.$$

Prove that

(a)

$$E(X) = \lim_{n\to\infty} E(X_n).$$

(b)

$$E(X^2) = \lim_{n\to\infty} E(X_n^2).$$

(c)

$$V(X) = \lim_{n\to\infty} V(X_n).$$

2. Let X_0 be a non-negative random variable (i.e., $P(X_0 \geq 0) = 1$), such that $E(X_0^2) < \infty$. Define

$$X_{n+1} = 6 + \sqrt{X_n}, \qquad n = 0, 1, 2, \ldots, .$$

Show that

$$\underset{n\to\infty}{\text{l.i.m.}}\ X_n = 9.$$

3. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables with mean zero, such that

$$E(X_i X_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Does the series

$$\sum_{k=1}^{n} \frac{X_k}{k}$$

converge in mean square as $n \to \infty$?

4. Show that if

$$X = \underset{n\to\infty}{\text{l.i.m.}}\ X_n, \qquad Y = \underset{n\to\infty}{\text{l.i.m.}}\ Y_n,$$

then

$$aX + bY = \underset{n\to\infty}{\text{l.i.m.}}\ (aX_n + bY_n)$$

for any constants a and b. Start from the definition and use suitable inequalities.

5. Let $\{Z_n\}_{n=-\infty}^{\infty}$ be a sequence of independent, identically distributed random variables such that $E(Z_n) = 0$ and $V(Z_n) = \sigma^2 < \infty$.

(a) Show that if $|c| < 1$, for each fixed n, the series

$$\sum_{i=0}^{m} c^i Z_{n-i}$$

converges in mean square as $m \to \infty$.

(b) Define for each $n \in \mathbb{Z}$ the random variable X_n by

$$X_n = \sum_{i=0}^{\infty} c^i Z_{n-i},$$

which is legitimate in view of (a) when $|c| < 1$. Show that the random variables X_n satisfy the stochastic difference equation

$$X_n = cX_{n-1} + Z_n \qquad \forall n \in \mathbb{Z}.$$

[Remark: we say that the process $\{X_n\}_{n=-\infty}^{\infty}$ is an autoregressive process of order 1, with acronym AR(1).]

(c) Compute the expectation $E(X_n)$ and the variance $V(X_n)$ using Theorem 4.1.

(d) Find the variance $V(X_n)$ without using the definition of X_n as the limit of a random series, but using the facts (to be proven later, in Lecture 8) that the process $\{X_n\}_{n=-\infty}^{\infty}$ is wide sense stationary, and that Z_n is independent of X_{n-k} for each $n \in \mathbb{Z}$ and $k \geq 1$.

6. A stochastic process $\{X(t); t \geq 0\}$ has mean value function $\mu_X = 0$, and autocorrelation function

$$R_X(t, \tau) = E(X(t)\, X(t+\tau)) = \sqrt{\min(t, t+\tau)}.$$

Is the process $\{X(t); t \geq 0\}$ mean square continuous?