Experimental Design and Biostatistics (TAMS38)
Lecture 6 – Factorial design, Latin Square Design

Martin Singull

Department of Mathematics
Mathematical Statistics
Linköping University, Sweden

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Factorial design

An experiment is affected by factors A, B, C with levels $A_1, \dots, A_a$ and $B_1, \dots, B_b$ and $C_1, \dots, C_c$, respectively.

For each level combination $A_i B_j C_k$ one obtains n measurements and obtain values y_{ijkl}.

Model: A complete three factor model is written as

$$Y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijkl} = \mu_{ijk} + \varepsilon_{ijkl}$$

where ε are independent and $N(0, \sigma)$,

$$\sum_{i=1}^{a}\tau_i = 0,\ \sum_{j=1}^{b}\beta_j = 0,\ \sum_{k=1}^{c}\gamma_k = 0,$$

$$\sum_{i=1}^{a}(\tau\beta)_{ij} = 0,\ \forall j,\ \sum_{j=1}^{b}(\tau\beta)_{ij} = 0\ \forall i,$$

$$\sum_{i=1}^{a}(\tau\gamma)_{ik} = 0\ \forall k,\ \sum_{k=1}^{c}(\tau\gamma)_{ik} = 0\ \forall i,$$

$$\sum_{j=1}^{b}(\beta\gamma)_{jk} = 0\ \forall k,\ \sum_{k=1}^{c}(\beta\gamma)_{jk} = 0\ \forall j,$$

$$\sum_{i=1}^{a}(\tau\beta\gamma)_{ijk} = 0 \forall j,k, \sum_{j=1}^{b}(\tau\beta\gamma)_{ijk} = 0 \forall i,k, \sum_{k=1}^{c}(\tau\beta\gamma)_{ijk} = 0 \forall i,j.$$

In total we have

$$1 + (a-1) + (b-1) + (c-1) + (a-1)(b-1) + (a-1)(c-1) + (b-1)(c-1) + (a-1)(b-1)(c-1) = abc$$

free mean parameters.

A complete three factor model can be also written as

$$Y_{ijkl} = \mu_{ijk} + \varepsilon_{ijkl},$$

where μ_{ijk} can vary freely.

This means that observations y_{ijkl} represents abc samples of size n and with different expected values.

Point estimators

$$\hat{\mu} = \bar{y}_{\cdots\cdot},$$

$$\hat{\tau}_i = \bar{y}_{i\cdots} - \bar{y}_{\cdots\cdot}, \quad \hat{\beta}_j = \bar{y}_{\cdot j\cdot\cdot} - \bar{y}_{\cdots\cdot}, \quad \hat{\gamma}_k = \bar{y}_{\cdot\cdot k\cdot} - \bar{y}_{\cdots\cdot},$$

$$\widehat{(\tau\beta)}_{ij} = \bar{y}_{ij\cdot\cdot} - \hat{\tau}_i - \hat{\beta}_j - \hat{\mu} = \bar{y}_{ij\cdot\cdot} - \bar{y}_{i\cdots} - \bar{y}_{\cdot j\cdot\cdot} + \bar{y}_{\cdots\cdot},$$

$$\widehat{(\tau\gamma)}_{ik} = \bar{y}_{i\cdot k\cdot} - \bar{y}_{i\cdots} - \bar{y}_{\cdot\cdot k\cdot} + \bar{y}_{\cdots\cdot},$$

$$\widehat{(\beta\gamma)}_{jk} = \bar{y}_{\cdot jk\cdot} - \bar{y}_{\cdot j\cdot\cdot} - \bar{y}_{\cdot\cdot k\cdot} + \bar{y}_{\cdots\cdot},$$

$$\begin{aligned}\widehat{(\tau\beta\gamma)}_{ijk} &= \bar{y}_{ijk\cdot} - \widehat{(\tau\beta)}_{ij} - \widehat{(\tau\gamma)}_{ik} - \widehat{(\beta\gamma)}_{jk} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\mu} \\ &= \underbrace{\bar{y}_{ijk\cdot}}_{=\hat{\mu}_{ijk}} - \bar{y}_{ij\cdot\cdot} - \bar{y}_{i\cdot k\cdot} - \bar{y}_{\cdot jk\cdot} + \bar{y}_{i\cdots} + \bar{y}_{\cdot j\cdot\cdot} + \bar{y}_{\cdot\cdot k\cdot} - \bar{y}_{\cdots\cdot}\end{aligned}$$

Excercise - Unbiasedness

Prove that the estimators given on previous slide are unbiased, i.e.,

$$E(\hat{\mu}) = E\left(\overline{Y}_{....}\right) = E\Big(\frac{1}{abcn}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{n}(\mu + \tau_i + \beta_j + \gamma_k$$

$$+ (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijkl})\Big)$$

$$= \frac{1}{abcn}\Big(abcn\mu + \sum_{j,k,l}\underbrace{\sum_{i}\tau_i}_{=0} + \sum_{i,k,l}\underbrace{\sum_{j}\beta_j}_{=0} + \sum_{i,j,l}\underbrace{\sum_{k}\gamma_k}_{=0}$$

$$+ \sum_{j,k,l}\underbrace{\sum_{i}(\tau\beta)_{ij}}_{=0} + \sum_{j,k,l}\underbrace{\sum_{i}(\tau\gamma)_{ik}}_{=0} + \sum_{i,k,l}\underbrace{\sum_{j}(\beta\gamma)_{jk}}_{=0}$$

$$+ \sum_{i,k,l}\underbrace{\sum_{j}(\tau\beta\gamma)_{ijk}}_{=0} + \sum_{i,j,k}\sum_{l}\underbrace{E(\varepsilon_{ijkl})}_{=0}\Big) = \mu.$$

Sum of squares

For the complete model we have

$$SS_A = nbc \sum_{i=1}^{a} \hat{\tau}_i^2, \quad df_A = a - 1$$

$$SS_B = nac \sum_{j=1}^{b} \hat{\beta}_j^2, \quad df_B = b - 1$$

$$SS_C = nab \sum_{k=1}^{c} \hat{\gamma}_k^2, \quad df_C = c - 1$$

$$SS_{AB} = nc \sum_{i=1}^{a} \sum_{j=1}^{b} \widehat{(\tau\beta)}_{ij}^2, \quad df_{AB} = (a-1)(b-1)$$

$$\vdots$$

$$SS_{ABC} = n \sum_i \sum_j \sum_k \widehat{(\tau\beta\gamma)}_{ijk}^2, \quad df_{ABC} = (a-1)(b-1)(c-1)$$

$$SS_E = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \sum_{l=1}^{n} (y_{ijkl} - \bar{y}_{ijk\cdot})^2 = \sum_{i,j,k} (n-1) s_{ijk}^2, \; df_E = abc(n-1)$$

Just as before, one can show that the sums of squares under certain conditions and after division by σ^2 are χ^2-distributed with the specified degrees of freedom.

Observe that degrees of freedom for SS_E are $abc(n-1)$ only in case of **complete** three-factor model.

If one uses additive model or some other specified model, one obtain some other degrees of freedom and new SS'_E.

For the **complete** model we have unbiased σ^2-estimator as

$$s^2 = \frac{SS_E}{abc(n-1)}$$

Furthermore, it holds that $\overline{Y}_{ijk\cdot}$ are independent of r.v. SS_E. This is important when one construct confidence interval.

F-test

For the complete model it holds that

(i) $H_{0A}:\ \tau_1 = \ldots = \tau_a = 0$ mot $H_{1A}:$ not all $\tau_i = 0$ is tested with help of

$$v_A = \frac{SS_A/(a-1)}{SS_E/[abc(n-1)]} \underset{H_{0A}}{\sim} F(a-1, abc(n-1)).$$

H_{0A} is rejected for the large values of v_A.

(ii) $H_{0AB} : (\tau\beta)_{ij} = 0\ \forall i, j$ (i.e. no interaction between A and B) against H_{1AB} : not all $(\tau\beta)_{ij} = 0$ (there is interaction between A and B) is tested with

$$v_{AB} = \frac{SS_{AB}/[(a-1)(b-1)]}{SS_E/[abc(n-1)]} \underset{H_{0AB}}{\sim} F((a-1)(b-1), abc(n-1)).$$

H_{0AB} is rejected for the large values of v_{AB}.

(iii) Other main effects and interaction effects are tested similarly.

Example 1

One wants to examine three factors that influence the production of a certain kind of units.

A: skill levels of staff.
Levels: A1, A2, A3 (decreasing)

B: the complexity of the method.
Levels: B1 (high), B2 (low)

C: tiredness.
Levels C1 (am), C2 (pm)

A full factorial design with two observations per cell was conducted and a observations y of quality and quantity were determined.

```
MTB > print Y
```

Data Display

```
Y
 92   88   88   84   78   82   78   77   70   74   69   71
 69   66   63   67   62   59   60   58   64   60   62   59

MTB > set c2
DATA> (1 2 3)8
DATA> end
MTB > set c3
DATA> 3(1 1 1 1 2 2 2 2)
DATA> end
MTB > set c4
DATA> 6(1 1 2 2)
DATA> end
```

```
MTB > anova Y=A|B|C.
```

ANOVA: Y versus A, B, C

Factor	Type	Levels	Values
A	fixed	3	1, 2, 3
B	fixed	2	1, 2
C	fixed	2	1, 2

Analysis of Variance for Y

Source	DF	SS	MS	F	P
A	2	2151.58	1075.79	195.60	0.000
B	1	104.17	104.17	18.94	0.001
C	1	32.67	32.67	5.94	0.031
A*B	2	116.58	58.29	10.60	0.002
A*C	2	3.08	1.54	0.28	0.760
B*C	1	0.17	0.17	0.03	0.865
A*B*C	2	1.08	0.54	0.10	0.907
Error	12	66.00	5.50		
Total	23	2475.33			

S = 2.34521   R-Sq = 97.33%   R-Sq(adj) = 94.89%

```
MTB > ANOVA 'Y' = A| B 'C';
SUBC>   Means A|B C.
```

ANOVA: Y versus A, B, C

Factor	Type	Levels	Values
A	fixed	3	1, 2, 3
B	fixed	2	1, 2
C	fixed	2	1, 2

Analysis of Variance for Y

Source	DF	SS	MS	F	P
A	2	2151.58	1075.79	260.03	0.000
B	1	104.17	104.17	25.18	0.000
A*B	2	116.58	58.29	14.09	0.000
C	1	32.67	32.67	7.90	0.012
Error	17	70.33	4.14		
Total	23	2475.33			

S = 2.03402   R-Sq = 97.16%   R-Sq(adj) = 96.16%

Analysis of Variance for Y=A|B|C

Source	DF	SS	MS	F	P
A	2	2151.58	1075.79	195.60	0.000
B	1	104.17	104.17	18.94	0.001
C	1	32.67	32.67	5.94	0.031
A*B	2	116.58	58.29	10.60	0.002
A*C	2	3.08	1.54	0.28	0.760
B*C	1	0.17	0.17	0.03	0.865
A*B*C	2	1.08	0.54	0.10	0.907
Error	12	66.00	5.50		
Total	23	2475.33			

Analysis of Variance for Y=A|B C

Source	DF	SS	MS	F	P
A	2	2151.58	1075.79	260.03	0.000
B	1	104.17	104.17	25.18	0.000
A*B	2	116.58	58.29	14.09	0.000
C	1	32.67	32.67	7.90	0.012
Error	17	70.33	4.14		
Total	23	2475.33			

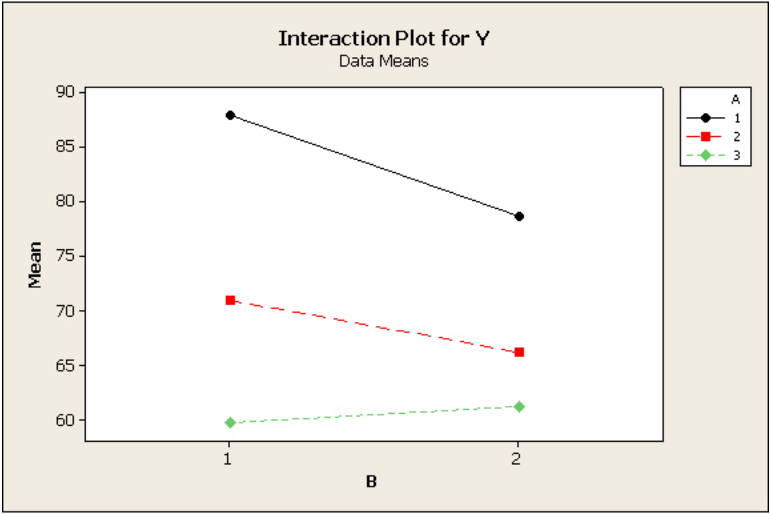
Means

A	N	Y
1	8	83.375
2	8	68.625
3	8	60.500

C	N	Y
1	12	72.000
2	12	69.667

B	N	Y
1	12	72.917
2	12	68.750

A	B	N	Y
1	1	4	88.000
1	2	4	78.750
2	1	4	71.000
2	2	4	66.250
3	1	4	59.750
3	2	4	61.250


Interaction Plot for Y
Data Means
Mean
90
85
80
75
70
65
60
1
2
B
A
1
2
3

The interaction plot shows that skilled workers (A1) seem to perform worse when they change from B1 to B2.

A3-persons obtain approximately the same results for B1 and B2.

Mean values for factor C suggest that there are better results in the morning than in the afternoon.

We analyses this using confidence intervals.

$\gamma_1 - \gamma_2$ is estimated with

$$\widehat{\gamma}_1 - \widehat{\gamma}_2 = (\overline{y}_{..1.} - \overline{y}_{....}) - (\overline{y}_{..2.} - \overline{y}_{....}) = \overline{y}_{..1.} - \overline{y}_{..2.}.$$

We have the r.v. $\overline{Y}_{..1.} - \overline{Y}_{..2.} \sim N\left(\gamma_1 - \gamma_2, \sqrt{\frac{\tilde{\sigma}^2}{12} + \frac{\tilde{\sigma}^2}{12}}\right)$ where

$\tilde{\sigma}^2$ is estimated with $\tilde{s}^2 = \frac{SS_E'}{17} = 4.137;\; s = 2.034;\; df_E = 17.$

$$\Rightarrow \quad \frac{\overline{Y}_{..1.} - \overline{Y}_{..2.} - (\gamma_1 - \gamma_2)}{\tilde{S}/\sqrt{6}} \sim t(17) \quad \text{and}$$

$$I_{\gamma_1 - \gamma_2} = \left(2.333 \mp 2.11 \cdot \frac{\tilde{s}}{\sqrt{6}}\right) \approx (0.58, 4.09) \;\; (95\%)$$

Only positive values. One has better production results in the morning.

To verify the trends seen on interaction plot, we construct intervals $I_{m_{i1}-m_{i2}}$ for $i = 1, 2, 3$. This means that we only make the interesting comparisons.

We have $\widehat{m}_{i1} - \widehat{m}_{i2} = \bar{y}_{i1\cdot\cdot} - \bar{y}_{i2\cdot\cdot}$

The r.v. $\overline{Y}_{i1\cdot\cdot} - \overline{Y}_{i2\cdot\cdot} \sim N\left(m_{i1} - m_{i2}, \sqrt{\frac{\tilde{\sigma}^2}{4} + \frac{\tilde{\sigma}^2}{4}}\right)$ gives

$$\Rightarrow \quad \frac{\overline{Y}_{i1\cdot\cdot} - \overline{Y}_{i2\cdot\cdot} - (m_{i1} - m_{i2})}{\tilde{S}/\sqrt{2}} \sim t(17) \quad \text{and}$$

$$I_{m_{i1}-m_{i2}} = \left(\bar{y}_{i1\cdot\cdot} - \bar{y}_{i2\cdot\cdot} \mp 2.11 \cdot \frac{\tilde{s}}{\sqrt{2}}\right) = (\bar{y}_{i1\cdot\cdot} - \bar{y}_{i2\cdot\cdot} \mp 3.035).$$

We obtain

$$I_{m_{11}-m_{12}} \approx (6.2, 12.3); \quad I_{m_{21}-m_{22}} \approx (1.7, 7.9)$$

and for A1 and A2 the complicated methods are better than the methods of low complexity.

$$I_{m_{31}-m_{32}} \approx (-4.5, 1.5),$$

i.e. no significant difference between B1 and B2 for A3.

Simultaneous confidence level $1 - \alpha_{sim} \geq 1 - 4 \cdot 0.05 = 0.80$ for those 4 intervals that we have constructed.

Note: In this example, the conclusions was obtained using the fact that we could simplify the model by disregarding some parameters. Such simplifications are therefore of great advantage, but they should be done with caution for eventual lose of information.

Block design, Latin square...

- Think carefully about which factors influence the results.
- Are these factors measurable?
- How can we take them into account when planning and analyzing the data?
- How can we eliminate or at least reduce the influence of possible disturbing factors?

Randomization $\Rightarrow$ Disturbing factors will at least not systematically influence.

Block $\Rightarrow$ Gives some control over the disturbing negative impact.

Example 2

One wants to compare the growth of four different varieties of wheat V_1, V_2, V_3, V_4. As design we dived a field into 16 areas:

Meadow

Forest

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Field

Lake

Complete randomized design

Choose randomly four boxes for V_1, four for V_2 etc.

V_1 : 04, 08, 15, 02 V_2 : 07, 01, 14, 05
V_3 : 13, 16, 11, 06 V_4 : 03, 09, 10, 12

Model 1. (One-Way ANOVA) Wheat type i gives yield y_{il} where

$$Y_{il} = \mu_i + \varepsilon_{il}$$

and $\varepsilon_{il} \sim N(0, \sigma)$ are assumed to be independent.

Here the 16 areas are considered to be equivalent. Any eventual differences between the areas "are added to" ε-variable. The mean parameters μ_i are a characteristic value of the wheat variety i.

Block design

If the rows on the field suspected to be different, but the areas in a row are equivalent, one put wheat so that every wheat variety occurs the same number of times in row.

The areas within the row are selected randomly for each wheat varieties. (Randomization in blocks = rows.)

$$\begin{matrix} V_2 & V_1 & V_3 & V_4 \\ V_3 & V_1 & V_2 & V_4 \\ V_3 & V_4 & V_1 & V_2 \\ V_4 & V_2 & V_1 & V_3 \end{matrix}$$

Model (Block design) Wheat type i and row j gives yield y_{ij} where

$$Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}$$

and $\sum_i \tau_i = 0, \sum_j \beta_j = 0$ and $\varepsilon \sim N(0, \sigma)$ are assumed to be independent.

Parameters τ_i describe effects of wheat variety and β_j effect of block, i.e., rows. We use to assume that block effect is additive. We compare different sorts of wheat by comparison of τ_i.

Latin square design

If there are differences both between rows and columns on the field we make a design according to a Latin square.

For a Latin square every wheat variety occurs exactly once in each row and exactly once in each column, i.e.,

$$\begin{matrix} V_2 & V_4 & V_3 & V_1 \\ V_4 & V_3 & V_1 & V_2 \\ V_3 & V_1 & V_2 & V_4 \\ V_1 & V_2 & V_4 & V_3 \end{matrix}$$

Model 3. (Latin square design) Wheat type i, row j and column k give yield $y_{ij(k)}$ where

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + \varepsilon_{ij(k)},$$

$\sum_i \tau_i = 0, \sum_j \beta_j = 0, \sum_k \gamma_k = 0$ och $\varepsilon_{ij(k)}$ is assumed to be independent and $N(0, \sigma)$.

The previous model has been extended with the parameter γ_k that takes care of the column properties.

We now have the following point estimates

$$\hat{\mu} = \bar{y}_{\ldots},$$
$$\hat{\tau}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\ldots},$$
$$\hat{\beta}_j = \bar{y}_{\cdot j\cdot} - \bar{y}_{\ldots},$$
$$\hat{\gamma}_k = \bar{y}_{\cdot\cdot k} - \bar{y}_{\ldots}.$$

Sums of squares

For a $p \times p$ - square we have

$$SS_A = p\sum_{i=1}^{p}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2, \quad df = p-1,$$
$$SS_B = p\sum_{j=1}^{p}(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})^2, \quad df = p-1,$$
$$SS_C = p\sum_{k=1}^{p}(\bar{y}_{\cdot\cdot k} - \bar{y}_{\cdot\cdot\cdot})^2, \quad df = p-1$$

and

$$SS_E = \sum_{i,j,(k)} (y_{ij(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k)^2$$
$$= \sum_{i,j,(k)} (y_{ij(k)} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot k} + 2\bar{y}_{\cdot\cdot\cdot})^2, \quad df = (p-2)(p-1).$$

F-test

With a design according to a Latin square we estimate the variance σ^2 by

$$s^2 = \frac{SS_E}{(p-2)(p-1)}$$

and test the hypothesis

$$H_{0A}: \ \tau_1 = \ldots = \tau_p = 0 \text{ (A-levels are equivalent)}$$

vs.

$$H_{1A}: \ \text{not all } \tau_i = 0$$

with the test statistics

$$v_A = \frac{SS_A/(p-1)}{SS_E/[(p-2)(p-1)]}.$$

The null hypothesis is rejected for large values of v_A.

Pairwise comparisons

One can do pairwise comparisons (of for example different kinds of wheat) through confidence intervals for $\tau_i - \tau_q$. Note that

$$\hat{\tau}_i - \hat{\tau}_q = \bar{y}_{i\cdot\cdot} - \bar{y}_{q\cdot\cdot}$$

after some simplification and the r.v.

$$\overline{Y}_{i\cdot\cdot} \sim N\left(\mu + \tau_i, \frac{\sigma}{\sqrt{p}}\right)$$

as it is average of p random variables each with variance σ^2.

The Latin square design can be extended to a designed called Graeco-Latin Square. See the Hand in Assignment 2.

Block design in relation to general factorial design

Ideally, each level combination occur as many times within each of the blocks. One **randomizes** within each block to prevent the systematic influence of disturbing factors such that example time.

Block effect is assumed to be **additive**. Reduced design where for example only some specified combinations of levels occur within blocks must be planned carefully, read about Roman squares, and see Lecture 7.

There are several methods other than those included in the course.

Remember: A design as to be combined with the "correct" model if we are to benefit from it.

Example 3 (Lecture 1)

The purpose of a study was to see if music during work effected the production in a factory.

Four different music programs A, B, C and D were compared with no music at all, E.

Each program was played for a day and one wanted five replicates for each program, i.e., the study lasted for five weeks.

Since there can also be variations between the days and between the weeks the design for the study was chosen to be a *Latin square*.

Results:

Week	Monday	Tuesday	Wednesday	Thursday	Friday
1	A 133	B 139	C 140	D 140	E 145
2	B 136	C 141	D 143	E 146	A 139
3	C 140	A 138	E 142	B 139	D 139
4	D 129	E 132	A 137	C 136	B 140
5	E 132	D 144	B 143	A 142	C 142

Note that each music program is present one time in each row and in each column.

Data are analyzed using Minitab, see below.

a) According to which model were data analyzed? Does the music seem to have no effect on production? Perform the appropriate test level 5%.

b) Is production worse on Mondays than on other days? Motivate your answer with use of confidence interval on the simultaneous confidence level at least 80%.
You can assume that even before you saw the result you suspected that production would be the worst on Mondays.

c) Does the normal distribution assumption seem to be reasonable? Explain your answer briefly.

```
MTB > GLM 'Y' = week day music;
SUBC>   Means week day music.
```

General Linear Model: Y versus week, day, music

```
Analysis of Variance for Y, using Adjusted SS for Tests

Source  DF  Seq SS  Adj SS  Adj MS     F      P
week    4  123.44  123.44   30.86  3.07  0.059
day      4  177.84  177.84   44.46  4.42  0.020
music    4   11.84   11.84    2.96  0.29  0.876
Error   12  120.72  120.72   10.06
Total   24  433.84

S = 3.17175   R-Sq = 72.17%   R-Sq(adj) = 44.35%
```

Least Squares Means for Y

week	Mean	SE Mean
1	139.4	1.418
2	141.0	1.418
3	139.6	1.418
4	134.8	1.418
5	140.6	1.418
day		
1	134.0	1.418
2	138.8	1.418
3	141.0	1.418
4	140.6	1.418
5	141.0	1.418
music		
1	137.8	1.418
2	139.4	1.418
3	139.8	1.418
4	139.0	1.418
5	139.4	1.418

a) Week i, day j, music k give observation $y_{ij(k)}$ where

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + \varepsilon_{ij(k)}$$

and $\varepsilon_{ij(k)} \sim N(0, \sigma)$.

We test $H_{0M}: \gamma_1 = \ldots = \gamma_k = 0$ vs. $H_{1M}:$ not all $\gamma_i = 0$, with an F-test.

Test statistics: $v_M = \dfrac{SS_M/4}{SS_E/12} = \dfrac{2.960}{10.060} = 0.29$

The r.v. $V_M \sim F(4, 12)$ if H_{0M} is true and v_M is large if H_{0M} is false.

Table gives the critical limit as $c = 3.26$ and $0.29 \ll 3.26$. The effect of music seems to be negligible.

b) We construct intervals $I_{\beta_j-\beta_1}$, downward limited and each of the confidence level 0.95=1-0.20/4. We have

$$\hat{\beta}_j - \hat{\beta}_1 = (\bar{y}_{\cdot j(\cdot)} - \bar{y}_{\cdot\cdot(\cdot)}) - (\bar{y}_{\cdot 1(\cdot)} - \bar{y}_{\cdot\cdot(\cdot)}) = \bar{y}_{\cdot j(\cdot)} - \bar{y}_{\cdot 1(\cdot)}$$

with corresponding r.v.

$$\overline{Y}_{\cdot j(\cdot)} - \overline{Y}_{\cdot 1(\cdot)} \sim N\left(\beta_j - \beta_1,\ \sigma\sqrt{\frac{2}{5}}\right)$$

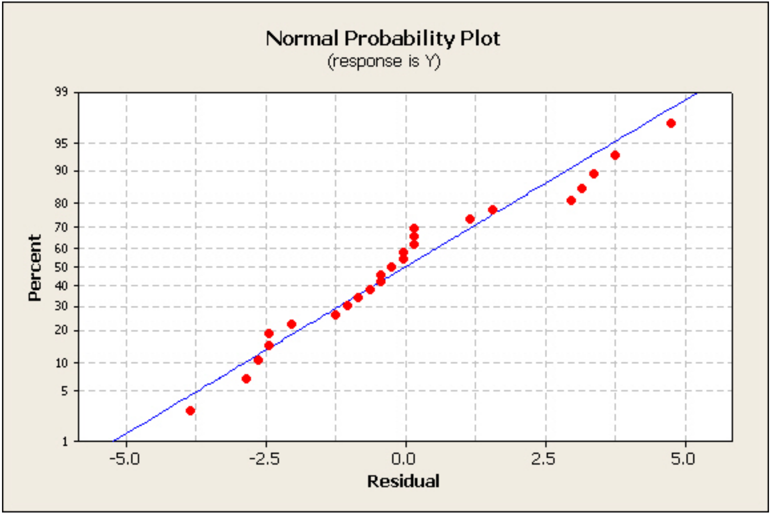
σ^2 is estimated with $s^2 = \dfrac{SS_E}{12} = 10.060;\ s = 3.172;\ df = 12$ and

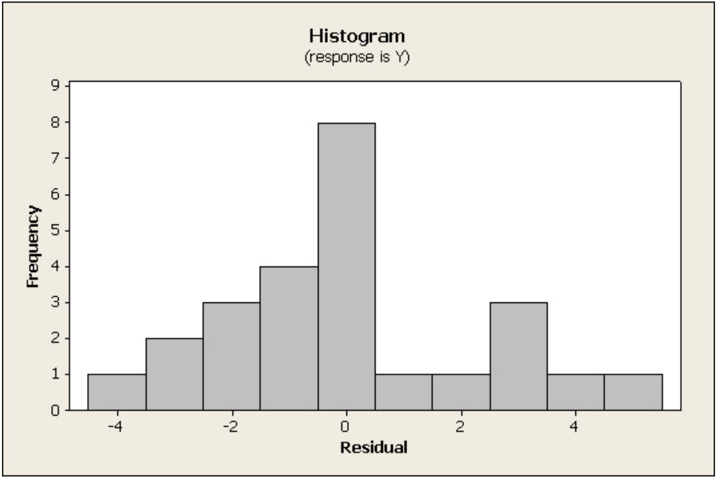
$$\frac{\overline{Y}_{\cdot j(\cdot)} - \overline{Y}_{\cdot 1(\cdot)} - (\beta_j - \beta_1)}{S \cdot \sqrt{2/5}} \sim t(12)$$

which gives

$$\begin{aligned} I_{\beta_2-\beta_1} &= (\bar{y}_{\cdot 2(\cdot)} - \bar{y}_{\cdot 1(\cdot)} - t \cdot s \cdot \sqrt{2/5},\ \infty) \\ &= (138.80 - 134.00 - 3.57,\ \infty) = (1.23,\ \infty), \quad \text{etc.} \end{aligned}$$

Production seems to be lower on Mondays.


Normal Probability Plot
(response is Y)
Percent
99
95
90
80
70
60
50
40
30
20
10
5
1
-5.0
-2.5
0.0
2.5
5.0
Residual



c) Neither the normal probability plot or histogram is really good, but the normal distribution can probably be good enough approximation.

Linköping University - Research that makes a difference

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