

Examiner: Xiangfeng Yang (Tel: 013-285788).

Things allowed (Hjälpmedel): a calculator.

Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.

1 (4 points)

Let (X, Y) be a two dimensional continuous random variable with a joint probability density function $f(x, y)$ given by

$$f(x, y) = x + y, \quad \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1.$$

(1.1) (1p) Find the probability $P(X + Y \leq 0.5)$.

(1.2) (1p) Find the conditional probability $P(X \leq 0.2 \mid Y \leq 0.6)$.

(1.3) (1p) Find the marginal probability density functions $f_X(x)$ of X and $f_Y(y)$ of Y .

(1.4) (1p) Find the covariance $cov(X, Y)$ which is defined as $cov(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$.

Solution. (1.1)

$$\begin{aligned} P(X + Y \leq 0.5) &= \int_0^{0.5} \left(\int_0^{0.5-x} (x + y) dy \right) dx = \int_0^{0.5} (1/8 - x^2/2) dx \\ &= [x/8 - x^3/6]_0^{0.5} = 1/24 = 0.042. \end{aligned}$$

(1.2)

$$\begin{aligned} P(X \leq 0.2 \mid Y \leq 0.6) &= \frac{P(X \leq 0.2 \cap Y \leq 0.6)}{P(Y \leq 0.6)} = \frac{\int_0^{0.2} \left(\int_0^{0.6} (x + y) dy \right) dx}{\int_0^1 \left(\int_0^{0.6} (x + y) dy \right) dx} \\ &= \frac{\int_0^{0.2} (0.6x + 0.18) dx}{\int_0^1 (0.6x + 0.18) dx} = 0.048/0.48 = 0.1. \end{aligned}$$

(1.3)

$$f_X(x) = \int_0^1 (x + y) dy = 0.5 + x, \quad 0 \leq x \leq 1, \quad f_Y(y) = \int_0^1 (x + y) dx = 0.5 + y, \quad 0 \leq y \leq 1.$$

(1.4)

$$\begin{aligned} cov(X, Y) &= E(X \cdot Y) - E(X) \cdot E(Y) = \int_0^1 \left(\int_0^1 xy(x + y) dy \right) dx - \int_0^1 x f_X(x) dx \cdot \int_0^1 y f_Y(y) dy \\ &= 1/3 - (7/12)^2 = -1/144 = -0.007. \end{aligned}$$

□

2 (3 points)

(2.1) (1p) Let $X_1 \sim N(6, 2^2)$. Find the probability $P(X_1 \leq 8)$.

(2.2) (1p) Let $X_2 \sim Bin(10, 0.5)$. Find the probability $P(3 \leq X_2 \leq 8)$.

(2.3) (1p) Let $X_3 \sim Po(1.6)$. Find the probability $P(3 < X_3 \leq 6)$.

Solution. (2.1)

$$P(X_1 \leq 8) = P\left(\frac{X_1 - 6}{2} \leq \frac{8 - 6}{2}\right) = P(N(0, 1) \leq 1) = \Phi(1) = 0.8413.$$

(2.2)

$$P(3 \leq X_2 \leq 8) = F(8) - F(2) = (\text{Binomial table}) = 0.9893 - 0.0547 = 0.9346.$$

(2.3)

$$P(3 < X_3 \leq 6) = F(6) - F(3) = (\text{Poisson table}) = 0.9987 - 0.9212 = 0.0775.$$

□

3 (2 points)

Two events A and B are independent if $P(A \cap B) = P(A) \cdot P(B)$. Now given two events A and B satisfying $P(A) > 0$ and $P(B|A) = P(B)$, do A and B have to be independent? If you think ‘yes’, then give a proof; if you think ‘no’, then give a counterexample.

Solution. Yes! If $P(B|A) = P(B)$, then A and B have to be independent. The proof is as follows (since $P(A) > 0$):

$$P(B|A) = P(B) \Rightarrow \frac{P(A \cap B)}{P(A)} = P(B) \Rightarrow P(A \cap B) = P(A) \cdot P(B).$$

□

4 (3 points)

A population $X \sim \text{Bin}(n, p)$ is a Binomial random variable with unknown parameters n and p . A sample $\{1, 3\}$ is observed from this population. Use the Method of Moments to find point estimates \hat{n}_{MM} and \hat{p}_{MM} of n and p .

Solution. It is from the method of moments that $E(X) = \bar{x} = \frac{x_1 + x_2}{2}$ and $E(X^2) = \frac{x_1^2 + x_2^2}{2}$, implying

$$np = \frac{x_1 + x_2}{2} = 2, \tag{1}$$

$$np(1-p) + (np)^2 = \frac{x_1^2 + x_2^2}{2} = 5, \tag{2}$$

where for (2) we have used the fact $V(X) = E(X^2) - (E(X))^2 \Rightarrow E(X^2) = V(X) + (E(X))^2 = np(1-p) + (np)^2$. Based on (1), we now replace every np in (2) by 2 and obtain

$$2(1-p) + 2^2 = 5 \Rightarrow \hat{p}_{MM} = 0.5.$$

Then (1) gives that $\hat{n}_{MM} = 2/\hat{p}_{MM} = 4$.

□

5 (3 points)

A company produces plenty of items, and according to the past data 60% of the items are classified as ‘high quality’ and 40% of the items are classified as ‘low quality’. Recently the company has implemented a new technology, and it is expected that the quality should be improved. Let p be the percentage of items with ‘high quality’ using this new technology, and one wants to test, with a significance level $\alpha = 0.05$,

$$H_0 : p = 60\% \quad \text{against} \quad H_a : p > 60\%.$$

In order to perform the hypotheses testing, a sample with sample size $n = 85$ is taken from the population and the sample percentage is $\hat{p} = 65\%$.

(5.1) (1p) Is H_0 rejected? Use TS and C method.

(5.2) (1p) Is H_0 rejected? Use p -value method.

(5.3) (1p) Find the power $h(0.75)$ when the real percentage is $p = 0.75$.

Solution. (5.1) The observed value of the test statistic and the rejection region are

$$TS = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.65 - 0.6}{\sqrt{0.6(1-0.6)/85}} = 0.941, \quad C = (z_\alpha, \infty) = (1.645, \infty).$$

Since $TS \notin C$, we do NOT reject H_0 .

Here we want to test the hypothesis: (5.2) The p -value is

$$p\text{-value} = P(N(0, 1) \geq TS) = P(N(0, 1) \geq 0.941) = 1 - \Phi(0.94) = 1 - 0.8264 = 0.1736.$$

As $p\text{-value} \not\leq \alpha$, we do NOT reject H_0 .

(5.3) The power is computed as follows:

$$\begin{aligned}
 h(0.75) &= P(H_0 \text{ is rejected when } p = 0.75) = P(TS \in C \text{ when } p = 0.75) = P\left(\frac{\hat{P} - p_0}{\sqrt{p_0(1-p_0)/n}} > 1.645 \text{ when } p = 0.75\right) \\
 &= P(\hat{P} > 1.645\sqrt{p_0(1-p_0)/n} + p_0 \text{ when } p = 0.75) = P\left(\frac{\hat{P} - p}{\sqrt{p(1-p)/n}} > \frac{(1.645\sqrt{p_0(1-p_0)/n} + p_0) - p}{\sqrt{p(1-p)/n}}\right) \\
 &= P(N(0, 1) > \frac{(1.645\sqrt{0.6(1-0.6)/85} + 0.6) - 0.75}{\sqrt{0.75(1-0.75)/85}}) = P(N(0, 1) > -1.33) \\
 &= 1 - \Phi(-1.3) = 1 - (1 - \Phi(1.33)) = \Phi(1.33) = 0.9082.
 \end{aligned}$$

□

6 (3 points)

Two populations $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent with known σ_1^2 and σ_2^2 . A sample $\{x_1, x_2, \dots, x_8\}$ is taken from X which gives a 95% confidence interval of μ_1 as $I_{\mu_1} = (4.36, 6.32)$. Another sample $\{y_1, y_2, \dots, y_5\}$ is taken from Y which gives a 95% confidence interval of μ_2 as $I_{\mu_2} = (7.22, 11.14)$. Find a 95% confidence interval of $\mu_1 - \mu_2$.

Solution. As σ_1^2 and σ_2^2 are known, the confidence intervals are

$$I_{\mu_1} = \bar{x} \mp z_{\alpha/2} \frac{\sigma_1}{\sqrt{8}} = (4.36, 6.32) \quad \text{and} \quad I_{\mu_2} = \bar{y} \mp z_{\alpha/2} \frac{\sigma_2}{\sqrt{5}} = (7.22, 11.14),$$

implying that (where $z_{\alpha/2} = z_{0.025} = 1.96$)

$$\bar{x} = 5.34, \quad \sigma_1 = 1.414; \quad \bar{y} = 9.18, \quad \sigma_2 = 2.236.$$

Thus the 95% confidence interval of $\mu_1 - \mu_2$ is

$$I_{\mu_1 - \mu_2} = (\bar{x} - \bar{y}) \mp z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{8} + \frac{\sigma_2^2}{5}} = (5.34 - 9.18) \mp 1.96 \sqrt{\frac{1.414^2}{8} + \frac{2.236^2}{5}} = (-6.031, -1.649).$$

□

1. Basic probability

- (1.1) Conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
 (1.2) Total probability $P(B) = \sum_{i=1}^k P(B|A_i)P(A_i)$ where $\{A_i\}$ are disjoint and $\cup_{i=1}^k A_i = S$.
 (1.3) Bayes' Theorem $P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^k P(B|A_i)P(A_i)}$ where $\{A_i\}$ are in (1.2).

2. Random variables (r.v.s)

- (2.1) Discrete r.v. X has a pmf $p(x) = P(X = x)$ satisfying $p(x) \geq 0$ and $\sum p(x_i) = 1$,

$$\begin{array}{c|cccc} X & x_1 & x_2 & \cdots & x_n & \cdots \\ \hline p(x) & p(x_1) & p(x_2) & \cdots & p(x_n) & \cdots \end{array}$$

- Expectation (or *Expected value* or *mean*) $\mu_X = E(X) = \sum x_i p(x_i)$;
 Variance $\sigma_X^2 = V(X) = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \sum x_i^2 p(x_i) - (\sum x_i p(x_i))^2$.
 (2.2) Continuous r.v. X has a pdf $f(x)$ satisfying $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x) dx = 1$,

$$P(a < X < b) = \int_a^b f(x) dx.$$

Expectation (or *Expected value* or *mean*) $\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$;

- Variance $\sigma_X^2 = V(X) = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - (\int_{-\infty}^{\infty} x f(x) dx)^2$.
 (2.3) Cumulative distribution function (cdf) of a r.v. X is $F(x) = P(X \leq x)$.
 (2.4) X and Y are r.v.s, a, b and c are scalars, then

$$E(aX + bY + c) = aE(X) + bE(Y) + c,$$

$$V(aX + bY + c) = a^2 V(X) + b^2 V(Y) + 2ab \operatorname{cov}(X, Y),$$

$$E(g(X, Y)) = \begin{cases} \sum_{i,j} g(x_i, y_j) \cdot p(x_i, y_j), & \text{for discrete } (X, Y), \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y) dx dy, & \text{for continuous } (X, Y). \end{cases}$$

- (2.5) • Discrete r.v. (X, Y) has a joint pmf $p(x, y)$ satisfying $p(x, y) \geq 0$ and $\sum_{x_i} \sum_{y_j} p(x_i, y_j) = 1$.
 The *marginal pmf* of X is $p_X(x) = \sum_y p(x, y)$;
 The *marginal pmf* of Y is $p_Y(y) = \sum_x p(x, y)$;
 X and Y are *independent* if $p(x, y) = p_X(x) \cdot p_Y(y)$.
 • Continuous r.v. (X, Y) has a joint pdf $f(x, y)$ satisfying $f(x, y) \geq 0$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$.
 The *marginal pdf* of X is $f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$;
 The *marginal pdf* of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$;
 X and Y are *independent* if $f(x, y) = f_X(x) \cdot f_Y(y)$.

3. Several special r.v.s

- (3.1) $X \sim \operatorname{Bin}(n, p)$ has a pmf $p(x) = P(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$, $x = 0, 1, 2, \dots, n$.
 $E(X) = n \cdot p$, $V(X) = n \cdot p \cdot (1-p)$.
 (3.2) $X \sim \operatorname{Po}(\lambda)$ has a pmf $p(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0, 1, 2, \dots$.
 $E(X) = \lambda$, $V(X) = \lambda$.

- (3.3) $X \sim \operatorname{Hypergeometric}$ has a pmf $p(x) = P(X = x) = \frac{\binom{M}{x} \binom{N-M}{n-x}}{\binom{N}{n}}$.

- (3.4) $X \sim \operatorname{Exp}(\lambda)$ has a pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

- (3.5) $X \sim N(\mu, \sigma^2)$ has a pdf

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

- (3.6) $X \sim U(a, b)$ has a pdf

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12}.$$

4. Central Limit Theorem (CLT)

Suppose that a population has mean $= \mu$ and variance $= \sigma^2$. A random sample $\{X_1, X_2, \dots, X_n\}$ from this population is given. Then for large $n \geq 30$,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1). \tag{1}$$

- If the population is normal, then (1) holds for any n .
- Note that $\mu = E(\bar{X})$ and $(\sigma/\sqrt{n})^2 = V(\bar{X})$.

5. Several notations in statistics

- (5.1) Sample mean: $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$; $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum x_i}{n}$.
 (5.2) Sample variance:

$$S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1} = \frac{1}{n-1} \left(\sum X_i^2 - \frac{(\sum X_i)^2}{n} \right); \quad s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = \frac{1}{n-1} \left(\sum x_i^2 - \frac{(\sum x_i)^2}{n} \right).$$

- Capital letters \bar{X} and S^2 refer to the objects based on random sample (therefore they are in general r.v.s), while small letters \bar{x} and s^2 are the objects based on observations (so they are scalars).
- (5.3) A point estimator of θ obtained by Methods of Moments is denoted as $\hat{\theta}_{MM}$.
- (5.4) A point estimator of θ obtained by Maximum Likelihood method is denoted as $\hat{\theta}_{ML}$.

6. Confidence Interval (CI)

In this course, three types of confidence intervals are studied depending on the unknown population parameter(s): CI-1 (confidence intervals for population mean(s)), CI-2 (confidence intervals for population variance(s)), and CI-3 (confidence intervals for population proportion(s)).

CI-1: (1 - α) CI of a population mean μ

case 1.1 (any n) If population $X \sim N(\mu, \sigma^2)$ and σ^2 is known, then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ and

$$I_\mu = (\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}) := \bar{x} \mp z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$

case 1.2 (n ≥ 30) For any population X , it holds that $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ and

$$I_\mu = \bar{x} \mp z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \text{ or } I_\mu = \bar{x} \mp z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}.$$

case 1.3 (any n) If population $X \sim N(\mu, \sigma^2)$ and σ^2 is unknown, then $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim T(n-1)$ and

$$I_\mu = \bar{x} \mp t_{\alpha/2}(n-1) \cdot \frac{s}{\sqrt{n}}.$$

CI-1': (1 - α) CI of the difference of two population means $\mu_X - \mu_Y$

case 1.1' (any n_1, n_2) If independent populations $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, and σ_X^2, σ_Y^2 are known, then

$$\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}} \sim N(0, 1), \text{ and } I_{\mu_X-\mu_Y} = (\bar{x}-\bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}.$$

case 1.2' ($n_1, n_2 \geq 30$) For any independent populations X and Y , it holds that

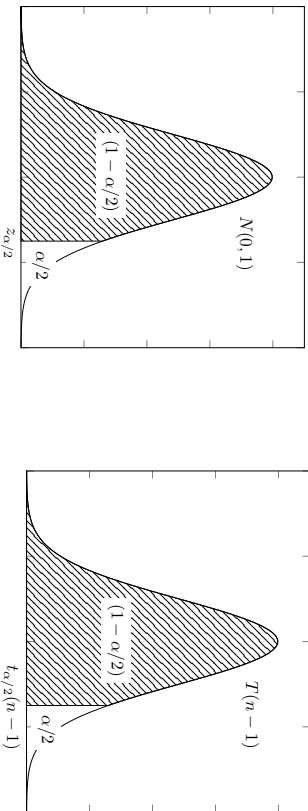
$$\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}} \sim N(0, 1) \text{ and}$$

$$I_{\mu_X-\mu_Y} = (\bar{x}-\bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}} \text{ or } I_{\mu_X-\mu_Y} = (\bar{x}-\bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{\sigma}_X^2}{n_1} + \frac{\hat{\sigma}_Y^2}{n_2}}.$$

case 1.3' (any n_1, n_2) If independent populations $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, where σ_X^2, σ_Y^2 are unknown but $\sigma_X^2 = \sigma_Y^2$, then

$$\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim T(n_1+n_2-2), \text{ where } S^2 = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2}{n_1+n_2-2}, \text{ and}$$

$$I_{\mu_X-\mu_Y} = (\bar{x}-\bar{y}) \mp t_{\alpha/2}(n_1+n_2-2) \cdot s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}.$$



CI-2: (1 - α) CI of population variance(s) σ^2

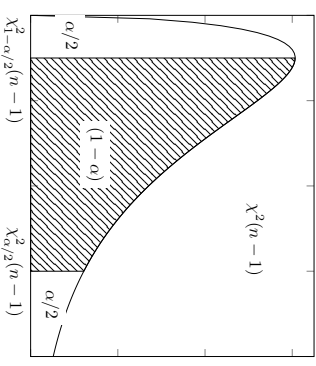
• If a population $X \sim N(\mu, \sigma^2)$ and σ^2 is unknown, then $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, and

$$I_{\sigma^2} = \left(\frac{(n-1)s^2}{\chi_{\alpha/2}^2(n-1)}, \frac{(n-1)s^2}{\chi_{1-\alpha/2}^2(n-1)} \right).$$

• If two independent populations $X \sim N(\mu_X, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma^2)$, and σ^2 is unknown, then $\frac{(n_1+n_2-2)S^2}{\sigma^2} \sim \chi^2(n_1+n_2-2)$, and

$$I_{\sigma^2} = \left(\frac{(n_1+n_2-2)s^2}{\chi_{\alpha/2}^2(n_1+n_2-2)}, \frac{(n_1+n_2-2)s^2}{\chi_{1-\alpha/2}^2(n_1+n_2-2)} \right),$$

where $S^2 = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2}{n_1+n_2-2}$.



CI-3: (1 - α) CI of population proportion(s)

• If a (large) population has an unknown proportion p , then $\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \sim N(0, 1)$ if $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$ with $\hat{p} = x/n$, and $I_p = \hat{p} \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

• If two independent (large) populations have unknown proportions p_1 and p_2 , then

$$\frac{(\hat{p}_1-\hat{p}_2)-(p_1-p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \sim N(0, 1)$$

if $n_i\hat{p}_i \geq 10$ and $n_i(1-\hat{p}_i) \geq 10$ for $i = 1, 2$, and $I_{p_1-p_2} = (\hat{p}_1-\hat{p}_2) \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$.

7. Hypothesis Test (HT)

	H_0 is true	H_0 is false and $\theta = \theta_1$
reject H_0	(type I error or significance level) α	(power) $h(\theta_1)$
don't reject H_0	$1 - \alpha$	(type II error) $\beta(\theta_1) = 1 - h(\theta_1)$

reject $H_0 \Leftrightarrow TS \in C \Leftrightarrow p\text{-value} < \alpha$

χ^2 tests for populations (non-parametric tests)

Suppose that for a random sample of a population X , the n elements of it are classified into k disjoint groups $A_i, 1 \leq i \leq k$. For each group $A_i, 1 \leq i \leq k$, suppose that there are $N_i, 1 \leq i \leq k$ elements inside. Let $p_i = P(A_i)$ assuming a given distribution of X . Note that $p_1 + p_2 + \dots + p_k = 1$ and $N_1 + N_2 + \dots + N_k = n$. One wants to test the hypotheses

$$H_0 : P(A_i) = p_i, \quad 1 \leq i \leq k, \quad H_a : P(A_i) \neq p_i \text{ for some } 1 \leq i \leq k.$$

If n is large in the sense that $np_i \geq 5$ for all $1 \leq i \leq k$, then the test statistic is

$$\sum_{i=1}^k \frac{(N_i - np_i)^2}{np_i} \approx \chi^2(k-1).$$

Therefore the observation of the test statistic is

$$TS = \sum_{i=1}^k \frac{(n_i - np_i)^2}{np_i}, \text{ where } n_i \text{ is the observation of } N_i, 1 \leq i \leq k.$$

For the critical region C , one can take (note that if H_0 is true, then TS should be close to zero)

$$C = (\chi_{\alpha}^2(k-1), \infty).$$

The conclusion would be $TS \in C \iff H_0$ is rejected.

8. Linear and logistic regression

(Multiple) linear regression: $Y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + \varepsilon$, $\varepsilon \sim N(0, \sigma^2)$.

- Y : response variable (which is normal r.v.), $\{x_1, \dots, x_k\}$: predictors (which are scalars).
- sample: $\{(x_{11}, \dots, x_{1k}; y_1), (x_{21}, \dots, x_{2k}; y_2), \dots, (x_{n1}, \dots, x_{nk}; y_n)\}$.
- how to estimate $\beta_j \approx \hat{\beta}_j$: least square method, that is, to minimize $\sum_{i=1}^n (y_i - \hat{y}_i)^2$, where the estimated (multiple) linear regression line \hat{y} is

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k.$$

- $\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim T(n-k-1)$, this helps determine whether or not the real $\beta_j = 0$?
- $\sigma^2 \approx \frac{SSE}{n-k-1}$, this gives an estimation of the size of the error.
- $R^2 = \frac{SSR}{SSY}$, this gives how well the model is (if $R^2 \approx 1$, then the model fits the sample very well).
- How to test $\beta_1 = \dots = \beta_k = 0$? Use the random variable $\frac{SSR/k}{SSE/(n-k-1)} \sim F(k, n-k-1)$.

Logistic regression: Let Y can only take 0 or 1 with $P(Y=1) = p$ and $P(Y=0) = 1-p$.

$$E(Y) = p(x_1, \dots, x_k) = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}.$$

- Y : response variable (which is Bernoulli r.v. $P(Y=1) = p$ and $P(Y=0) = 1-p$, so $E(Y) = p$), $\{x_1, \dots, x_k\}$: predictors (which are scalars).
- sample: $\{(x_{11}, \dots, x_{1k}; y_1), (x_{21}, \dots, x_{2k}; y_2), \dots, (x_{n1}, \dots, x_{nk}; y_n)\}$.
- how to estimate $\beta_j \approx \hat{\beta}_j$: maximal likelihood method (maximize $\prod_{i=1}^n p(x_{i1}, \dots, x_{ik})^{y_i} (1 - p(x_{i1}, \dots, x_{ik}))^{1-y_i}$).
- $\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \approx N(0, 1)$ for large $n \geq 30$, this helps determine whether or not the real $\beta_j = 0$?
- Classification of a new object $Y(x_1, \dots, x_k)$ as 1 or 0 according

$$Y(x_1, \dots, x_k) = \begin{cases} 1, & \text{if } \hat{p}(x_1, \dots, x_k) \geq 0.5, \\ 0, & \text{if } \hat{p}(x_1, \dots, x_k) < 0.5, \end{cases}$$

where the estimated logit function $\hat{p}(x_1, \dots, x_k)$ is

$$\hat{p}(x_1, \dots, x_k) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k}}.$$

9. Tables

(9.1) Table for $N(0, 1)$ standard normal random variable $\Phi(x) = P(N(0, 1) \leq x)$, $x \geq 0$.
There is an important relation $\Phi(-x) = 1 - \Phi(x)$, $x \geq 0$.

x	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9564	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9993	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(9.2) Table for $T(f)$ random variable $F(x) = P(T(f) \leq x)$,
where f is a parameter called 'degrees of freedom'.

f	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.9995
1	1.00	3.08	6.31	12.71	31.82	63.66	127.32	636.62
2	0.82	1.89	2.92	4.30	6.96	9.92	14.09	31.60
3	0.76	1.64	2.35	3.18	4.54	5.84	7.45	12.92
4	0.74	1.53	2.13	2.78	3.75	4.60	5.60	8.61
5	0.73	1.48	2.02	2.57	3.36	4.03	4.77	6.87
6	0.72	1.44	1.94	2.45	3.14	3.71	4.32	5.96
7	0.71	1.41	1.89	2.36	3.00	3.50	4.03	5.41
8	0.71	1.40	1.86	2.31	2.90	3.36	3.83	5.04
9	0.70	1.38	1.83	2.26	2.82	3.25	3.69	4.78
10	0.70	1.37	1.81	2.23	2.76	3.17	3.58	4.59
11	0.70	1.36	1.80	2.20	2.72	3.11	3.50	4.44
12	0.70	1.36	1.78	2.18	2.68	3.05	3.43	4.32
13	0.69	1.35	1.77	2.16	2.65	3.01	3.37	4.22
14	0.69	1.35	1.76	2.14	2.62	2.98	3.33	4.14
15	0.69	1.34	1.75	2.13	2.60	2.95	3.29	4.07
16	0.69	1.34	1.75	2.12	2.58	2.92	3.25	4.01
17	0.69	1.33	1.74	2.11	2.57	2.90	3.22	3.97
18	0.69	1.33	1.73	2.10	2.55	2.88	3.20	3.92
19	0.69	1.33	1.73	2.09	2.54	2.86	3.17	3.88
20	0.69	1.33	1.72	2.09	2.53	2.85	3.15	3.85
21	0.69	1.32	1.72	2.08	2.52	2.83	3.14	3.82
22	0.69	1.32	1.72	2.07	2.51	2.82	3.12	3.79
23	0.69	1.32	1.71	2.07	2.50	2.81	3.10	3.77
24	0.68	1.32	1.71	2.06	2.49	2.80	3.09	3.75
25	0.68	1.32	1.71	2.06	2.49	2.79	3.08	3.73
26	0.68	1.31	1.71	2.06	2.48	2.78	3.07	3.71
27	0.68	1.31	1.70	2.05	2.47	2.77	3.06	3.69
28	0.68	1.31	1.70	2.05	2.47	2.76	3.05	3.67
29	0.68	1.31	1.70	2.05	2.46	2.76	3.04	3.66
30	0.68	1.31	1.70	2.04	2.46	2.75	3.03	3.65
40	0.68	1.30	1.68	2.02	2.42	2.70	2.97	3.55
50	0.68	1.30	1.68	2.01	2.40	2.68	2.94	3.50
60	0.68	1.30	1.67	2.00	2.39	2.66	2.91	3.46
100	0.68	1.29	1.66	1.98	2.36	2.63	2.87	3.39
∞	0.67	1.28	1.65	1.96	2.33	2.58	2.81	3.29

(9.3) Table for $\chi^2(f)$ random variable $F(x) = P(\chi^2(f) \leq x)$, where f is a parameter.

f	0.0005	0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.30	0.40	0.50
1	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.06	0.15	0.27	0.45
2	0.00	0.00	0.01	0.02	0.05	0.10	0.21	0.45	0.71	1.02	1.39
3	0.02	0.02	0.07	0.11	0.22	0.35	0.58	1.01	1.42	1.87	2.37
4	0.06	0.09	0.21	0.30	0.48	0.71	1.06	1.65	2.19	2.75	3.36
5	0.16	0.21	0.41	0.55	0.83	1.15	1.61	2.34	3.00	3.66	4.35
6	0.30	0.38	0.68	0.87	1.24	1.64	2.20	3.07	3.83	4.57	5.35
7	0.48	0.60	0.99	1.24	1.69	2.17	2.73	3.82	4.67	5.49	6.35
8	0.71	0.86	1.34	1.65	2.18	2.73	3.49	4.59	5.53	6.42	7.34
9	0.97	1.15	1.73	2.09	2.70	3.33	4.17	5.38	6.39	7.36	8.34
10	1.26	1.48	2.16	2.56	3.25	3.94	4.87	6.18	7.27	8.30	9.34
11	1.59	1.83	2.60	3.05	3.82	4.57	5.58	6.99	8.15	9.24	10.34
12	1.93	2.21	3.07	3.57	4.40	5.23	6.30	7.81	9.03	10.18	11.34
13	2.31	2.62	3.57	4.11	5.01	5.89	7.04	8.63	9.93	11.13	12.34
14	2.70	3.04	4.07	4.66	5.63	6.57	7.79	9.47	10.82	12.08	13.34
15	3.11	3.48	4.60	5.23	6.26	7.26	8.55	10.31	11.72	13.03	14.34
16	3.54	3.94	5.14	5.81	6.91	7.96	9.31	11.15	12.62	13.98	15.34
17	3.98	4.42	5.70	6.41	7.56	8.67	10.09	12.00	13.53	14.94	16.34
18	4.44	4.90	6.26	7.01	8.23	9.39	10.86	12.86	14.44	15.89	17.34
19	4.91	5.41	6.84	7.63	8.91	10.12	11.65	13.72	15.35	16.85	18.34
20	5.40	5.92	7.43	8.26	9.59	10.85	12.44	14.58	16.27	17.81	19.34
21	5.90	6.45	8.03	8.90	10.28	11.59	13.24	15.44	17.18	18.77	20.34
22	6.40	6.98	8.64	9.54	10.98	12.34	14.04	16.31	18.10	19.73	21.34
23	6.92	7.53	9.26	10.20	11.69	13.09	14.85	17.19	19.02	20.69	22.34
24	7.45	8.08	9.89	10.86	12.40	13.85	15.66	18.06	19.94	21.65	23.34
25	7.99	8.65	10.52	11.52	13.12	14.61	16.47	18.94	20.87	22.62	24.34
26	8.54	9.22	11.16	12.20	13.84	15.38	17.29	19.82	21.79	23.58	25.34
27	9.09	9.80	11.81	12.88	14.57	16.15	18.11	20.70	22.72	24.54	26.34
28	9.66	10.39	12.46	13.56	15.31	16.93	18.94	21.59	23.65	25.51	27.34
29	10.23	10.99	13.12	14.26	16.05	17.71	19.77	22.48	24.58	26.48	28.34
30	10.80	11.59	13.79	14.95	16.79	18.49	20.60	23.36	25.51	27.44	29.34
40	16.91	17.92	20.71	22.16	24.43	26.51	29.05	32.34	34.87	37.13	39.34
50	23.46	24.67	27.99	29.71	32.36	34.76	37.69	41.45	46.86	51.93	59.33
60	30.34	31.74	35.53	37.48	40.48	43.19	46.46	50.64	53.81	56.62	59.33
100	59.90	61.92	67.33	70.06	74.22	77.93	82.36	87.95	92.13	95.81	99.33

Table for $\chi^2(f)$ random variable $F(x) = P(\chi^2(f) \leq x)$, where f is a parameter.

f	0.60	0.70	0.80	0.90	0.95	0.975	0.99	0.995	0.999	0.9995
1	0.71	1.07	1.64	2.71	3.84	5.02	6.63	7.88	10.83	12.12
2	1.83	2.41	3.22	4.61	5.99	7.38	9.21	10.60	13.82	15.20
3	2.95	3.66	4.64	6.25	7.81	9.35	11.34	12.84	16.27	17.73
4	4.04	4.88	5.99	7.78	9.49	11.14	13.28	14.86	18.47	20.00
5	5.13	6.06	7.29	9.24	11.07	12.83	15.09	16.75	20.52	22.11
6	6.21	7.23	8.56	10.64	12.59	14.45	16.81	18.55	22.46	24.10
7	7.28	8.38	9.80	12.02	14.07	16.01	18.48	20.28	24.32	26.02
8	8.35	9.52	11.03	13.36	15.51	17.53	20.09	21.95	26.12	27.87
9	9.41	10.66	12.24	14.68	16.92	19.02	21.67	23.59	27.88	29.67
10	10.47	11.78	13.44	15.99	18.31	20.48	23.21	25.19	29.59	31.42
11	11.53	12.90	14.63	17.28	19.68	21.92	24.72	26.76	31.26	33.14
12	12.58	14.01	15.81	18.55	21.03	23.34	26.22	28.30	32.91	34.82
13	13.64	15.12	16.98	19.81	22.36	24.74	27.69	29.82	34.53	36.48
14	14.69	16.22	18.15	21.06	23.68	26.12	29.14	31.32	36.12	38.11
15	15.73	17.32	19.31	22.31	25.00	27.49	30.58	32.80	37.70	39.72
16	16.78	18.42	20.47	23.54	26.30	28.85	32.00	34.27	39.25	41.31
17	17.82	19.51	21.61	24.77	27.59	30.19	33.41	35.72	40.79	42.88
18	18.87	20.60	22.76	25.99	28.87	31.53	34.81	37.16	42.31	44.43
19	19.91	21.69	23.90	27.20	30.14	32.85	36.19	38.58	43.82	45.97
20	20.95	22.77	25.04	28.41	31.41	34.17	37.57	40.00	45.31	47.50
21	21.99	23.86	26.17	29.62	32.67	35.48	38.93	41.40	46.80	49.01
22	23.03	24.94	27.30	30.81	33.92	36.78	40.29	42.80	48.27	50.51
23	24.07	26.02	28.43	32.01	35.17	38.08	41.64	44.18	49.73	52.00
24	25.11	27.10	29.55	33.20	36.42	39.36	42.98	45.56	51.18	53.48
25	26.14	28.17	30.68	34.38	37.65	40.65	44.31	46.93	52.62	54.95
26	27.18	29.25	31.79	35.56	38.89	41.92	45.64	48.29	54.05	56.41
27	28.21	30.32	32.91	36.74	40.11	43.19	46.96	49.64	55.48	57.86
28	29.25	31.39	34.03	37.92	41.34	44.46	48.28	50.99	56.89	59.30
29	30.28	32.46	35.14	39.09	42.56	45.72	49.59	52.34	58.30	60.73
30	31.32	33.53	36.25	40.26	43.77	46.98	50.89	53.67	59.70	62.16
40	41.62	44.16	47.27	51.81	55.76	59.34	63.69	66.77	73.40	76.09
50	51.89	54.72	58.16	63.17	67.50	71.42	76.15	79.49	86.66	89.56
60	62.13	65.23	68.97	74.40	79.08	83.30	88.38	91.95	99.61	102.69
100	102.95	106.91	111.67	118.50	124.34	129.56	135.81	140.17	149.45	153.17

(9.4) Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
 If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1 - p) \geq n - k)$.

n	k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
2	0	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225	0.3600	0.3025	0.2500
	1	0.9975	0.9900	0.9775	0.9600	0.9375	0.9100	0.8775	0.8400	0.7975	0.7500
3	0	0.8574	0.7290	0.6141	0.5120	0.4219	0.3430	0.2746	0.2160	0.1664	0.1250
	1	0.9928	0.9720	0.9392	0.8960	0.8438	0.7840	0.7183	0.6480	0.5747	0.5000
4	0	0.8145	0.6561	0.5220	0.4096	0.3164	0.2401	0.1785	0.1256	0.0915	0.0625
	1	0.9860	0.9477	0.8905	0.8192	0.7383	0.6517	0.5630	0.4735	0.3910	0.3125
5	0	0.7738	0.5905	0.4437	0.3277	0.2373	0.1681	0.1160	0.0778	0.0503	0.0313
	1	0.9774	0.9185	0.8352	0.7373	0.6328	0.5282	0.4284	0.3370	0.2562	0.1875
6	0	0.6972	0.8857	0.7765	0.6554	0.5339	0.4202	0.3191	0.2333	0.1636	0.1094
	1	0.9978	0.9914	0.9734	0.9421	0.8965	0.8369	0.7648	0.6826	0.5931	0.5000
7	0	0.6383	0.4783	0.3206	0.2097	0.1335	0.0824	0.0490	0.0280	0.0152	0.0078
	1	0.9566	0.8503	0.7166	0.5767	0.4449	0.3294	0.2338	0.1586	0.1024	0.0625
8	0	0.5634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039
	1	0.9428	0.8131	0.6572	0.5033	0.3671	0.2553	0.1691	0.1064	0.0632	0.0352
9	0	0.4928	0.7748	0.5995	0.4362	0.3003	0.1960	0.1211	0.0705	0.0385	0.0195
	1	0.9916	0.9470	0.8591	0.7382	0.6007	0.4628	0.3373	0.2318	0.1495	0.0898

Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
 If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1 - p) \geq n - k)$.

n	k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
10	0	0.5987	0.3487	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010
	1	0.9139	0.7361	0.5443	0.3758	0.2440	0.1493	0.0860	0.0464	0.0233	0.0107
11	0	0.5688	0.3138	0.1673	0.0859	0.0422	0.0198	0.0088	0.0036	0.0014	0.0005
	1	0.8981	0.6974	0.4922	0.3221	0.1971	0.1130	0.0606	0.0302	0.0139	0.0059
12	0	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057	0.0022	0.0008	0.0002
	1	0.8816	0.6590	0.4435	0.2749	0.1584	0.0850	0.0424	0.0196	0.0083	0.0032

(9.5) Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	μ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725	0.7358
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371	0.9197
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9875	0.9810
4	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977	0.9963
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9994
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	μ									
	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019	0.1827	0.1653	0.1496	0.1353
1	0.6990	0.6626	0.6268	0.5918	0.5578	0.5249	0.4932	0.4628	0.4337	0.4060
2	0.9004	0.8795	0.8571	0.8335	0.8088	0.7834	0.7572	0.7306	0.7037	0.6767
3	0.9743	0.9662	0.9569	0.9463	0.9344	0.9212	0.9068	0.8913	0.8747	0.8571
4	0.9946	0.9923	0.9893	0.9857	0.9814	0.9763	0.9704	0.9636	0.9559	0.9473
5	0.9990	0.9985	0.9978	0.9968	0.9955	0.9940	0.9920	0.9896	0.9868	0.9834
6	0.9999	0.9997	0.9996	0.9994	0.9991	0.9987	0.9981	0.9974	0.9966	0.9955
7	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9996	0.9994	0.9992	0.9989
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	μ									
	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	0.1225	0.1108	0.1003	0.0907	0.0821	0.0743	0.0672	0.0608	0.0550	0.0498
1	0.3796	0.3546	0.3309	0.3084	0.2873	0.2674	0.2487	0.2311	0.2146	0.1991
2	0.6496	0.6227	0.5960	0.5697	0.5438	0.5184	0.4936	0.4695	0.4460	0.4232
3	0.8386	0.8194	0.7993	0.7787	0.7576	0.7360	0.7141	0.6919	0.6696	0.6472
4	0.9379	0.9275	0.9162	0.9041	0.8912	0.8774	0.8629	0.8477	0.8318	0.8153
5	0.9796	0.9751	0.9700	0.9643	0.9580	0.9510	0.9433	0.9349	0.9258	0.9161
6	0.9941	0.9925	0.9906	0.9884	0.9858	0.9828	0.9794	0.9756	0.9713	0.9665
7	0.9985	0.9980	0.9974	0.9967	0.9958	0.9947	0.9934	0.9919	0.9901	0.9881
8	0.9997	0.9995	0.9994	0.9991	0.9989	0.9985	0.9981	0.9976	0.9969	0.9962
9	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996	0.9995	0.9993	0.9991	0.9989
10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	μ									
	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
0	0.0408	0.0334	0.0273	0.0224	0.0183	0.0150	0.0123	0.0101	0.0082	0.0067
1	0.1712	0.1468	0.1257	0.1074	0.0916	0.0780	0.0663	0.0563	0.0477	0.0404
2	0.3799	0.3397	0.3027	0.2689	0.2381	0.2102	0.1851	0.1626	0.1425	0.1247
3	0.6025	0.5584	0.5152	0.4735	0.4335	0.3954	0.3594	0.3257	0.2942	0.2650
4	0.7806	0.7442	0.7064	0.6678	0.6288	0.5898	0.5512	0.5132	0.4763	0.4405
5	0.8946	0.8705	0.8441	0.8156	0.7851	0.7531	0.7199	0.6858	0.6510	0.6160
6	0.9534	0.9421	0.9267	0.9091	0.8893	0.8675	0.8436	0.8180	0.7908	0.7622
7	0.9832	0.9769	0.9682	0.9599	0.9509	0.9419	0.9321	0.9214	0.9049	0.8867
8	0.9943	0.9917	0.9883	0.9840	0.9786	0.9721	0.9642	0.9549	0.9442	0.9319
9	0.9982	0.9973	0.9960	0.9942	0.9919	0.9889	0.9851	0.9805	0.9749	0.9682
10	0.9995	0.9992	0.9987	0.9981	0.9972	0.9959	0.9943	0.9922	0.9896	0.9863
11	0.9999	0.9998	0.9996	0.9994	0.9991	0.9986	0.9980	0.9971	0.9960	0.9945
12	1.0000	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9980
13	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9996	0.9995	0.9994	0.9993
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
k	μ									
	5.2	5.4	5.6	5.8	6.0	6.5	7.0	7.5	8.0	8.5
0	0.0055	0.0045	0.0037	0.0030	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002
1	0.0342	0.0289	0.0244	0.0206	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019
2	0.1088	0.0948	0.0824	0.0715	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093
3	0.2381	0.2133	0.1906	0.1700	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301
4	0.4061	0.3733	0.3422	0.3127	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744
5	0.5809	0.5461	0.5119	0.4783	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496
6	0.7324	0.7017	0.6703	0.6384	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562
7	0.8449	0.8217	0.7970	0.7710	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856
8	0.9181	0.9027	0.8857	0.8672	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231
9	0.9603	0.9512	0.9409	0.9292	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530
10	0.9823	0.9775	0.9718	0.9649	0.9574	0.9408	0.9243	0.9015	0.8819	0.8634
11	0.9927	0.9904	0.9875	0.9841	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487
12	0.9972	0.9962	0.9949	0.9932	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091
13	0.9990	0.9986	0.9980	0.9973	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486
14	0.9999	0.9995	0.9993	0.9990	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726
15	0.9999	0.9998	0.9998	0.9996	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862
16	1.0000	0.9999	0.9999	0.9999	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934
17	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970
18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9993	0.9987
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9995
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	μ														
	9.0	9.5	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0					
0	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000					
1	0.0012	0.0008	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000					
2	0.0062	0.0042	0.0028	0.0012	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000					
3	0.0212	0.0149	0.0103	0.0049	0.0023	0.0011	0.0005	0.0002	0.0001	0.0000					
4	0.0550	0.0403	0.0293	0.0151	0.0076	0.0037	0.0018	0.0009	0.0004	0.0002					
5	0.1157	0.0885	0.0671	0.0375	0.0203	0.0107	0.0055	0.0028	0.0014	0.0007					
6	0.2068	0.1649	0.1301	0.0786	0.0458	0.0259	0.0142	0.0076	0.0040	0.0021					
7	0.3239	0.2687	0.2202	0.1432	0.0895	0.0540	0.0316	0.0180	0.0100	0.0054					
8	0.4557	0.3918	0.3328	0.2320	0.1550	0.0998	0.0621	0.0374	0.0220	0.0126					
9	0.5874	0.5218	0.4579	0.3405	0.2424	0.1658	0.1094	0.0699	0.0433	0.0261					
10	0.7060	0.6453	0.5830	0.4599	0.3472	0.2517	0.1757	0.1185	0.0774	0.0491					
11	0.8030	0.7520	0.6968	0.5793	0.4616	0.3532	0.2600	0.1848	0.1270	0.0847					
12	0.8758	0.8364	0.7916	0.6887	0.5760	0.4631	0.3585	0.2676	0.1931	0.1350					
13	0.9261	0.8981	0.8645	0.7813	0.6815	0.5730	0.4644	0.3632	0.2745	0.2009					
14	0.9585	0.9400	0.9165	0.8540	0.7720	0.6751	0.5704	0.4657	0.3675	0.2808					
15	0.9780	0.9665	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681	0.4667	0.3715					
16	0.9889	0.9823	0.9730	0.9441	0.8987	0.8355	0.7559	0.6641	0.5660	0.4677					
17	0.9947	0.9911	0.9857	0.9678	0.9370	0.8905	0.8272	0.7489	0.6593	0.5640					
18	0.9976	0.9957	0.9928	0.9823	0.9626	0.9302	0.8826	0.8195	0.7423	0.6550					
19	0.9989	0.9980	0.9965	0.9907	0.9787	0.9573	0.9235	0.8752	0.8122	0.7363					
20	0.9996	0.9991	0.9984	0.9953	0.9884	0.9750	0.9521	0.9170	0.8682	0.8055					
21	0.9998	0.9996	0.9993	0.9977	0.9939	0.9859	0.9712	0.9469	0.9108	0.8615					
22	0.9999	0.9999	0.9997	0.9990	0.9970	0.9924	0.9833	0.9673	0.9418	0.9047					
23	1.0000	0.9999	0.9999	0.9995	0.9985	0.9960	0.9907	0.9805	0.9633	0.9367					
24	1.0000	1.0000	1.0000	0.9998	0.9993	0.9980	0.9950	0.9888	0.9777	0.9594					
25	1.0000	1.0000	1.0000	0.9999	0.9997	0.9990	0.9974	0.9938	0.9869	0.9748					
26	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9987	0.9967	0.9925	0.9848					
27	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9983	0.9959	0.9912					
28	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9978	0.9950					
29	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9994	0.9986					
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993					
31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9996					
32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9996					
33	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999					
34	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999					
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000					