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Things allowed (Hjälpmedel): a calculator.

Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.

1 (3 points)

In a Swedish town, suppose that 55% people have subscribed the newspaper “Aftonbladet”, 60% people have subscribed the newspaper “Expressen”, and 31% people have subscribed both newspapers. Now we randomly ask one person in the town.

(1.1) (1p) Find the probability that the person has subscribed the newspaper “Aftonbladet”, but not the newspaper “Expressen”.

(1.2) (1p) Find the probability that the person has subscribed at least one of the newspapers “Aftonbladet” and “Expressen”.

(1.3) (1p) Given the person has subscribed the newspaper “Aftonbladet”, what is the probability the person has also subscribed the newspaper “Expressen”?

Solution. Define two events:

$$A = \{\text{one person has subscribed “Aftonbladet”}\}, \quad E = \{\text{one person has subscribed “Expressen”}\}.$$

Then we know that $P(A) = 55\%$, $P(E) = 60\%$ and $P(A \cap E) = 31\%$.

(1.1)

$$\begin{aligned} &P(\text{the person has subscribed the newspaper “Aftonbladet”, but not the the newspaper “Expressen”}) \\ &= P(A \cap E') = (\text{draw a Venn diagram to see this}) = P(A) - P(A \cap E) = 55\% - 31\% = 24\%. \end{aligned}$$

(1.2)

$$\begin{aligned} &P(\text{the person has subscribed at least one of the newspapers “Aftonbladet” and “Expressen”}) \\ &= P(A \cup E) = P(A) + P(E) - P(A \cap E) = 55\% + 60\% - 31\% = 84\%. \end{aligned}$$

(1.3) This is a conditional probability, and it is

$$P(E|A) = \frac{P(E \cap A)}{P(A)} = \frac{31\%}{55\%} = 56.36\%.$$

□

2 (3 points)

We are going to measure an object n times and take the average of these n measurements as our final measurement. For each measurement i , $1 \leq i \leq n$, there might be an error X_i , $1 \leq i \leq n$. Suppose that these n errors X_1, X_2, \dots, X_n are independent and have the same distribution which is continuous with a probability density function as follows

$$f(x) = \begin{cases} 0.5, & \text{if } -1 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now we consider the average error $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$. How large should n be in order that $P(|\bar{X}| \leq 0.1) = 0.95$? (namely, the probability of the average error being no more than 0.1 is 0.95).

Solution. Note that for a random variable X with the given probability density function $f(x)$, the mean and variance are

$$\mu = E(X) = \int_{-1}^1 xf(x)dx = 0, \quad \sigma^2 = V(X) = \int_{-1}^1 x^2 f(x)dx - 0^2 = 1/3.$$

Suppose that $n \geq 30$, then it is from CLT that

$$\begin{aligned}
0.95 &= P(|\bar{X}| \leq 0.1) = P(-0.1 \leq \bar{X} \leq 0.1) = P\left(\frac{-0.1 - \mu}{\sigma/\sqrt{n}} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{0.1 - \mu}{\sigma/\sqrt{n}}\right) \\
&= P\left(\frac{-0.1 - 0}{\sqrt{1/3}/\sqrt{n}} \leq N(0, 1) \leq \frac{0.1 - 0}{\sqrt{1/3}/\sqrt{n}}\right) \\
&= P\left(N(0, 1) \leq \frac{0.1}{\sqrt{1/(3n)}}\right) - P\left(N(0, 1) < -\frac{0.1}{\sqrt{1/(3n)}}\right) \\
&= \Phi\left(\frac{0.1}{\sqrt{1/(3n)}}\right) - \Phi\left(-\frac{0.1}{\sqrt{1/(3n)}}\right) = \Phi\left(\frac{0.1}{\sqrt{1/(3n)}}\right) - \left[1 - \Phi\left(\frac{0.1}{\sqrt{1/(3n)}}\right)\right] \\
&= 2 \cdot \Phi\left(\frac{0.1}{\sqrt{1/(3n)}}\right) - 1 \Rightarrow \Phi\left(\frac{0.1}{\sqrt{1/(3n)}}\right) = (1 + 0.95)/2 = 0.975.
\end{aligned}$$

It is from the standard normal table that $\Phi(a) = 0.975$ gives $a = 1.96$. Therefore,

$$\frac{0.1}{\sqrt{1/(3n)}} = 1.96 \Rightarrow n = 128.$$

□

3 (3 points)

Let (X, Y) be a two dimensional discrete random variable with a joint probability mass function $p(x, y)$ as follows

		Y	
		5	3
X	1	0.3	0.1
	2	0.2	0.4

The above table says that X can take values 1 and 2, and Y can take values 5 and 3.

(3.1) (1p) Find the mean $E(X)$ of X .

(3.2) (1p) Find the conditional probability $P(X = 2|Y = 5)$.

(3.3) (1p) Find the covariance $cov(X, Y)$ (which is defined as $cov(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$).

Solution. Two marginal tables can be directly obtained from the joint table as follows

X	1	2		Y	5	3
$p_X(x)$	0.4	0.6		$p_Y(y)$	0.5	0.5

(3.1) It is directly from the marginal table of X that

$$E(X) = 1 \cdot 0.4 + 2 \cdot 0.6 = 1.6.$$

(3.2)

$$P(X = 2|Y = 5) = \frac{P(X = 2, Y = 5)}{P(Y = 5)} = \frac{0.2}{0.5} = 0.4.$$

(3.3)

$$\begin{aligned}
cov(X, Y) &= E(X \cdot Y) - E(X) \cdot E(Y) \\
&= (1 \cdot 5 \cdot 0.3 + 1 \cdot 3 \cdot 0.1 + 2 \cdot 5 \cdot 0.2 + 2 \cdot 3 \cdot 0.4) - 1.6 \cdot 4 \\
&= 6.2 - 6.4 = -0.2.
\end{aligned}$$

□

4 (3 points)

Suppose that a huge box contains plenty of balls which are either white or black. Let p denote the proportion of white balls in the box. In order to estimate p , one randomly takes out 66 balls one by one with replacement (namely, one takes a ball, observes and records the color, then puts the ball back into the box, and takes another ball). Out of these 66 balls, there are 14 white balls.

(4.1) (1p) Estimate p by \hat{p}_{MM} using the Method of Moments.

(4.2) (2p) Estimate p by \hat{p}_{ML} using the Maximum-Likelihood method.

Solution. Let X be the number of white balls in the chosen 66 balls, then $X \sim \text{Bin}(66, p)$. One observation is $x = 14$.

(4.1) For MM, we use the first equation $E(X) = \bar{x}$ and obtain $66 \cdot p = x$ (note that here $\bar{x} = x/1 = x$ since we have only one observation). Therefore $\hat{p}_{MM} = x/66 = 14/66 = 21.21\%$.

(4.2) For ML, the likelihood function is

$$L(p) = p(x) = \binom{66}{x} \cdot p^x \cdot (1-p)^{66-x},$$

and the function after taking logarithm becomes

$$\ln L(p) = \ln \binom{66}{x} + x \cdot \ln p + (66 - x) \cdot \ln(1 - p).$$

Then

$$0 = \ln' L(p) = x/p - (66 - x)/(1 - p) \implies \hat{p}_{ML} = x/66 = 14/66 = 21.21\%.$$

□

5 (3 points)

Suppose that there is a population $X \sim N(\mu, \sigma^2)$. A sample is taken from this population with sample size 16, sample mean 2.6 and sample standard deviation 1.6.

(5.1) (1p) Construct a 95% two-sided confidence interval for μ .

(5.2) (2p) Construct a 95% one-sided confidence interval for σ^2 in the form $(a, +\infty)$.

Solution. (5.1) What we know are: $n = 16$, $\bar{x} = 2.6$ and $s = 1.6$. Since σ^2 is unknown, it follows that

$$I_\mu = \bar{x} \mp t_{\alpha/2}(n-1) \cdot \frac{s}{\sqrt{n}} = 2.6 \mp 2.13 \cdot \frac{1.6}{\sqrt{16}} = 2.6 \mp 0.852 = (1.748, 3.452).$$

(5.2) The one-sided confidence interval is

$$I_{\sigma^2} = \left(\frac{(n-1)s^2}{\chi_\alpha^2(n-1)}, +\infty \right) = \left(\frac{(16-1) \cdot 1.6^2}{25.00}, +\infty \right) = \left(\frac{38.4}{25.00}, +\infty \right) = (1.536, +\infty).$$

□

6 (3 points)

Let X be the distribution of weights of a certain (huge number of) products, and assume that $X \sim N(\mu, 3^2)$. A sample is taken from this population with $n = 25$, $\bar{x} = 5.3$ and $s = 3.1$. We want to test the following hypotheses

$$H_0 : \mu = 5, \quad H_a : \mu > 5,$$

with a significance level $\alpha = 5\%$.

(6.1) (1p) Use TS and C to perform the hypothesis test.

(6.2) (1p) Use p -value to perform the hypothesis test.

(6.3) (1p) Find the Type-II error when the true value is $\mu = 5.5$.

Solution. (6.1) Since here σ^2 is known, one needs to use the case 1.1. Therefore,

$$TS = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{5.3 - 5}{3/\sqrt{25}} = 0.5, \quad C = (z_\alpha, +\infty) = (1.64, +\infty),$$

and $TS \notin C$, which implies that H_0 is not rejected.

(6.2) In this case, one can calculate the p -value as

$$p\text{-value} = P(UL \geq TS) = P(N(0, 1) \geq 0.5) = 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085.$$

Since $p\text{-value} \not\leq \alpha$, we do not reject H_0 .

(6.3) The Type-II error is computed as follows

$$\begin{aligned} \beta(5.5) &= P(H_0 \text{ is NOT rejected when } H_0 \text{ is false and } \mu = 5.5) = P(TS \notin C, \text{ when } \mu = 5.5) \\ &= P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq 1.64, \text{ when } \mu = 5.5\right) \\ &= P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq 1.64 + \frac{(\mu_0 - \mu)}{\sigma/\sqrt{n}}, \text{ when } \mu = 5.5\right) \\ &= P(N(0, 1) \leq 0.81) = \Phi(0.81) = 0.7910. \end{aligned}$$

□

1. Basic probability

(1.1) Conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

(1.2) Total probability $P(B) = \sum_{i=1}^k P(B|A_i)P(A_i)$ where $\{A_i\}$ are disjoint and $\cup_{i=1}^k A_i = S$.

(1.3) Bayes' Theorem $P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^k P(B|A_i)P(A_i)}$ where $\{A_i\}$ are in (1.2).

2. Random variables (r.v.s)

(2.1) Discrete r.v. X has a pmf $p(x) = P(X = x)$ satisfying $p(x) \geq 0$ and $\sum p(x_i) = 1$,

$$\begin{array}{c|cccc} X & x_1 & x_2 & \cdots & x_n & \cdots \\ \hline p(x) & p(x_1) & p(x_2) & \cdots & p(x_n) & \cdots \end{array}$$

Expectation (or *Expected value* or *mean*) $\mu_X = E(X) = \sum x_i p(x_i)$;

Variance $\sigma_X^2 = V(X) = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \sum x_i^2 p(x_i) - (\sum x_i p(x_i))^2$.

(2.2) Continuous r.v. X has a pdf $f(x)$ satisfying $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x) dx = 1$,

$$P(a < X < b) = \int_a^b f(x) dx.$$

Expectation (or *Expected value* or *mean*) $\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$;

Variance $\sigma_X^2 = V(X) = E(X - \mu_X)^2 = E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - (\int_{-\infty}^{\infty} x f(x) dx)^2$.

(2.3) Cumulative distribution function (cdf) of a r.v. X is $F(x) = P(X \leq x)$.

(2.4) X and Y are r.v.s, a, b and c are scalars, then

$$E(aX + bY + c) = aE(X) + bE(Y) + c,$$

$$V(aX + bY + c) = a^2V(X) + b^2V(Y) + 2ab \operatorname{cov}(X, Y),$$

$$E(g(X, Y)) = \begin{cases} \sum_{i,j} g(x_i, y_j) \cdot p(x_i, y_j), & \text{for discrete } (X, Y), \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y) dx dy, & \text{for continuous } (X, Y). \end{cases}$$

(2.5) • Discrete r.v. (X, Y) has a joint pmf $p(x, y)$ satisfying $p(x, y) \geq 0$ and $\sum_{x_i} \sum_{y_j} p(x_i, y_j) = 1$.

The *marginal pmf* of X is $p_X(x) = \sum_y p(x, y)$;

The *marginal pmf* of Y is $p_Y(y) = \sum_x p(x, y)$;

X and Y are *independent* if $p(x, y) = p_X(x) \cdot p_Y(y)$.

• Continuous r.v. (X, Y) has a joint pdf $p(x, y)$ satisfying $f(x, y) \geq 0$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$.

The *marginal pdf* of X is $f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$;

The *marginal pdf* of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$;

X and Y are *independent* if $f(x, y) = f_X(x) \cdot f_Y(y)$.

3. Several special r.v.s

(3.1) $X \sim \operatorname{Bin}(n, p)$ has a pmf $p(x) = P(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$, $x = 0, 1, 2, \dots, n$.

$$E(X) = n \cdot p, \quad V(X) = n \cdot p \cdot (1-p).$$

(3.2) $X \sim \operatorname{Po}(\lambda)$ has a pmf $p(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0, 1, 2, \dots$
 $E(X) = \lambda, \quad V(X) = \lambda$.

(3.3) $X \sim \operatorname{Hypergeometric}$ has a pmf $p(x) = P(X = x) = \frac{\binom{M}{x} \binom{N-M}{n-x}}{\binom{N}{n}}$.

(3.4) $X \sim \operatorname{Exp}(\lambda)$ has a pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$E(X) = \frac{1}{\lambda}, \quad V(X) = \left(\frac{1}{\lambda}\right)^2$.

(3.5) $X \sim N(\mu, \sigma^2)$ has a pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

$E(X) = \mu, \quad V(X) = \sigma^2$.

(3.6) $X \sim U(a, b)$ has a pdf

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12}$.

4. Central Limit Theorem (CLT)

Suppose that a population has mean $= \mu$ and variance $= \sigma^2$. A random sample $\{X_1, X_2, \dots, X_n\}$ from this population is given. Then for large $n \geq 30$,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1). \tag{1}$$

• If the population is normal, then (1) holds for any n .

• Note that $\mu = E(\bar{X})$ and $(\sigma/\sqrt{n})^2 = V(\bar{X})$.

5. Several notations in statistics

(5.1) Sample mean: $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n} = \sum \frac{X_i}{n}$; $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \sum \frac{x_i}{n}$.

(5.2) Sample variance:

$$S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1} = \frac{1}{n-1} \left(\sum X_i^2 - \frac{(\sum X_i)^2}{n} \right); \quad s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = \frac{1}{n-1} \left(\sum x_i^2 - \frac{(\sum x_i)^2}{n} \right).$$

• Capital letters \bar{X} and S^2 refer to the objects based on random sample (therefore they are in general r.v.s), while small letters \bar{x} and s^2 are the objects based on observations (so they are scalars).

(5.3) A point estimator of θ obtained by Methods of Moments is denoted as $\hat{\theta}_{MM}$.

(5.4) A point estimator of θ obtained by Maximum Likelihood method is denoted as $\hat{\theta}_{ML}$.

6. Confidence Interval (CI)

In this course, three types of confidence intervals are studied depending on the unknown population parameter(s): CI-1 (confidence intervals for population mean(s)), CI-2 (confidence intervals for population variance(s)), and CI-3 (confidence intervals for population proportion(s)).

CI-1: (1 - α) CI of a population mean μ

case 1.1 (any n) If population $X \sim N(\mu, \sigma^2)$ and σ^2 is known, then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ and

$$I_\mu = (\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}) := \bar{x} \mp z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$

case 1.2 (n ≥ 30) For any population X , it holds that $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ and

$$I_\mu = \bar{x} \mp z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \text{ or } I_\mu = \bar{x} \mp z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}.$$

case 1.3 (any n) If population $X \sim N(\mu, \sigma^2)$ and σ^2 is unknown, then $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim T(n-1)$ and

$$I_\mu = \bar{x} \mp t_{\alpha/2}(n-1) \cdot \frac{s}{\sqrt{n}}.$$

CI-1': (1 - α) CI of the difference of two population means $\mu_X - \mu_Y$

case 1.1' (any n_1, n_2) If independent populations $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, and σ_X^2, σ_Y^2 are known, then

$$\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}} \sim N(0, 1), \text{ and } I_{\mu_X-\mu_Y} = (\bar{x}-\bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}.$$

case 1.2' ($n_1, n_2 \geq 30$) For any independent populations X and Y , it holds that

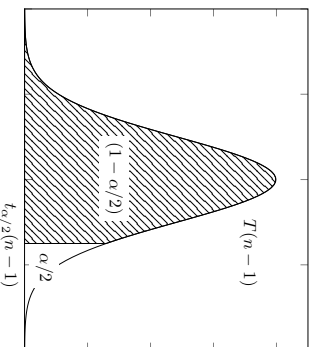
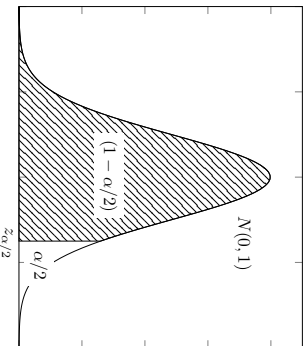
$$\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}} \sim N(0, 1) \text{ and}$$

$$I_{\mu_X-\mu_Y} = (\bar{x}-\bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}} \text{ or } I_{\mu_X-\mu_Y} = (\bar{x}-\bar{y}) \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{\sigma}_X^2}{n_1} + \frac{\hat{\sigma}_Y^2}{n_2}}.$$

case 1.3' (any n_1, n_2) If independent populations $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, where σ_X^2, σ_Y^2 are unknown but $\sigma_X^2 = \sigma_Y^2$, then

$$\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim T(n_1+n_2-2), \text{ where } S^2 = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2}{n_1+n_2-2}, \text{ and}$$

$$I_{\mu_X-\mu_Y} = (\bar{x}-\bar{y}) \mp t_{\alpha/2}(n_1+n_2-2) \cdot s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}.$$



CI-2: (1 - α) CI of population variance(s) σ^2

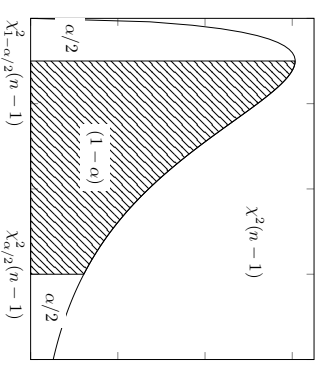
• If a population $X \sim N(\mu, \sigma^2)$ and σ^2 is unknown, then $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, and

$$I_{\sigma^2} = \left(\frac{(n-1)s^2}{\chi_{\alpha/2}^2(n-1)}, \frac{(n-1)s^2}{\chi_{1-\alpha/2}^2(n-1)} \right).$$

• If two independent populations $X \sim N(\mu_X, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma^2)$, and σ^2 is unknown, then $\frac{(n_1+n_2-2)S^2}{\sigma^2} \sim \chi^2(n_1+n_2-2)$, and

$$I_{\sigma^2} = \left(\frac{(n_1+n_2-2)s^2}{\chi_{\alpha/2}^2(n_1+n_2-2)}, \frac{(n_1+n_2-2)s^2}{\chi_{1-\alpha/2}^2(n_1+n_2-2)} \right),$$

where $S^2 = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2}{n_1+n_2-2}$.



CI-3: (1 - α) CI of population proportion(s)

• If a (large) population has an unknown proportion p , then $\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \sim N(0, 1)$ if $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$ with $\hat{p} = x/n$, and $I_p = \hat{p} \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

• If two independent (large) populations have unknown proportions p_1 and p_2 , then

$$\frac{(\hat{p}_1-\hat{p}_2)-(p_1-p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \sim N(0, 1)$$

if $n_i\hat{p}_i \geq 10$ and $n_i(1-\hat{p}_i) \geq 10$ for $i = 1, 2$, and $I_{p_1-p_2} = (\hat{p}_1-\hat{p}_2) \mp z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$.

7. Hypothesis Test (HT)

	H_0 is true	H_0 is false and $\theta = \theta_1$
reject H_0	(type I error or significance level) α	(power) $h(\theta_1)$
don't reject H_0	$1 - \alpha$	(type II error) $\beta(\theta_1) = 1 - h(\theta_1)$

reject $H_0 \Leftrightarrow TS \in C \Leftrightarrow p\text{-value} < \alpha$

χ^2 tests for populations (non-parametric tests)

Suppose that for a random sample of a population X , the n elements of it are classified into k disjoint groups $A_i, 1 \leq i \leq k$. For each group $A_i, 1 \leq i \leq k$, suppose that there are $N_i, 1 \leq i \leq k$ elements inside. Let $p_i = P(A_i)$ assuming a given distribution of X . Note that $p_1 + p_2 + \dots + p_k = 1$ and $N_1 + N_2 + \dots + N_k = n$. One wants to test the hypotheses

$$H_0 : P(A_i) = p_i, \quad 1 \leq i \leq k, \quad H_a : P(A_i) \neq p_i \text{ for some } 1 \leq i \leq k.$$

If n is large in the sense that $np_i \geq 5$ for all $1 \leq i \leq k$, then the test statistic is

$$\sum_{i=1}^k \frac{(N_i - np_i)^2}{np_i} \approx \chi^2(k-1).$$

Therefore the observation of the test statistic is

$$TS = \sum_{i=1}^k \frac{(n_i - np_i)^2}{np_i}, \text{ where } n_i \text{ is the observation of } N_i, 1 \leq i \leq k.$$

For the critical region C , one can take (note that if H_0 is true, then TS should be close to zero)

$$C = (\chi^2_{\alpha}(k-1), \infty).$$

The conclusion would be $TS \in C \iff H_0$ is rejected.

8. Linear and logistic regression

(Multiple) linear regression: $Y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + \varepsilon$, $\varepsilon \sim N(0, \sigma^2)$.

- Y : response variable (which is normal r.v.), $\{x_1, \dots, x_k\}$: predictors (which are scalars).
- sample: $\{(x_{11}, \dots, x_{1k}; y_1), (x_{21}, \dots, x_{2k}; y_2), \dots, (x_{n1}, \dots, x_{nk}; y_n)\}$.
- how to estimate $\beta_j \approx \hat{\beta}_j$: least square method, that is, to minimize $\sum_{i=1}^n (y_i - \hat{y}_i)^2$, where the estimated (multiple) linear regression line \hat{y} is

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k.$$

- $\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim T(n-k-1)$, this helps determine whether or not the real $\beta_j = 0$?
- $\sigma^2 \approx \frac{SSE}{n-k-1}$, this gives an estimation of the size of the error.
- $R^2 = \frac{SSR}{SSY}$, this gives how well the model is (if $R^2 \approx 1$, then the model fits the sample very well).
- How to test $\beta_1 = \dots = \beta_k = 0$? Use the random variable $\frac{SSR/k}{SSE/(n-k-1)} \sim F(k, n-k-1)$.

Logistic regression: Let Y can only take 0 or 1 with $P(Y=1) = p$ and $P(Y=0) = 1-p$.

$$E(Y) = p(x_1, \dots, x_k) = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}.$$

- Y : response variable (which is Bernoulli r.v. $P(Y=1) = p$ and $P(Y=0) = 1-p$, so $E(Y) = p$), $\{x_1, \dots, x_k\}$: predictors (which are scalars).
- sample: $\{(x_{11}, \dots, x_{1k}; y_1), (x_{21}, \dots, x_{2k}; y_2), \dots, (x_{n1}, \dots, x_{nk}; y_n)\}$.
- how to estimate $\beta_j \approx \hat{\beta}_j$: maximal likelihood method (maximize $\prod_{i=1}^n p(x_{i1}, \dots, x_{ik})^{y_i} (1 - p(x_{i1}, \dots, x_{ik}))^{1-y_i}$).
- $\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \approx N(0, 1)$ for large $n \geq 30$, this helps determine whether or not the real $\beta_j = 0$?
- Classification of a new object $Y(x_1, \dots, x_k)$ as 1 or 0 according

$$Y(x_1, \dots, x_k) = \begin{cases} 1, & \text{if } \hat{p}(x_1, \dots, x_k) \geq 0.5, \\ 0, & \text{if } \hat{p}(x_1, \dots, x_k) < 0.5, \end{cases}$$

where the estimated logit function $\hat{p}(x_1, \dots, x_k)$ is

$$\hat{p}(x_1, \dots, x_k) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \dots + \hat{\beta}_k x_k}}.$$

9. Tables

(9.1) Table for $N(0, 1)$ standard normal random variable $\Phi(x) = P(N(0, 1) \leq x)$, $x \geq 0$.
There is an important relation $\Phi(-x) = 1 - \Phi(x)$, $x \geq 0$.

x	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9564	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9993	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(9.2) Table for $T(f)$ random variable $F(x) = P(T(f) \leq x)$,
where f is a parameter called 'degrees of freedom'.

f	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.9995
1	1.00	3.08	6.31	12.71	31.82	63.66	127.32	636.62
2	0.82	1.89	2.92	4.30	6.96	9.92	14.09	31.60
3	0.76	1.64	2.35	3.18	4.54	5.84	7.45	12.92
4	0.74	1.53	2.13	2.78	3.75	4.60	5.60	8.61
5	0.73	1.48	2.02	2.57	3.36	4.03	4.77	6.87
6	0.72	1.44	1.94	2.45	3.14	3.71	4.32	5.96
7	0.71	1.41	1.89	2.36	3.00	3.50	4.03	5.41
8	0.71	1.40	1.86	2.31	2.90	3.36	3.83	5.04
9	0.70	1.38	1.83	2.26	2.82	3.25	3.69	4.78
10	0.70	1.37	1.81	2.23	2.76	3.17	3.58	4.59
11	0.70	1.36	1.80	2.20	2.72	3.11	3.50	4.44
12	0.70	1.36	1.78	2.18	2.68	3.05	3.43	4.32
13	0.69	1.35	1.77	2.16	2.65	3.01	3.37	4.22
14	0.69	1.35	1.76	2.14	2.62	2.98	3.33	4.14
15	0.69	1.34	1.75	2.13	2.60	2.95	3.29	4.07
16	0.69	1.34	1.75	2.12	2.58	2.92	3.25	4.01
17	0.69	1.33	1.74	2.11	2.57	2.90	3.22	3.97
18	0.69	1.33	1.73	2.10	2.55	2.88	3.20	3.92
19	0.69	1.33	1.73	2.09	2.54	2.86	3.17	3.88
20	0.69	1.33	1.72	2.09	2.53	2.85	3.15	3.85
21	0.69	1.32	1.72	2.08	2.52	2.83	3.14	3.82
22	0.69	1.32	1.72	2.07	2.51	2.82	3.12	3.79
23	0.69	1.32	1.71	2.07	2.50	2.81	3.10	3.77
24	0.68	1.32	1.71	2.06	2.49	2.80	3.09	3.75
25	0.68	1.32	1.71	2.06	2.49	2.79	3.08	3.73
26	0.68	1.31	1.71	2.06	2.48	2.78	3.07	3.71
27	0.68	1.31	1.70	2.05	2.47	2.77	3.06	3.69
28	0.68	1.31	1.70	2.05	2.47	2.76	3.05	3.67
29	0.68	1.31	1.70	2.05	2.46	2.76	3.04	3.66
30	0.68	1.31	1.70	2.04	2.46	2.75	3.03	3.65
40	0.68	1.30	1.68	2.02	2.42	2.70	2.97	3.55
50	0.68	1.30	1.68	2.01	2.40	2.68	2.94	3.50
60	0.68	1.30	1.67	2.00	2.39	2.66	2.91	3.46
100	0.68	1.29	1.66	1.98	2.36	2.63	2.87	3.39
∞	0.67	1.28	1.65	1.96	2.33	2.58	2.81	3.29

(9.3) Table for $\chi^2(f)$ random variable $F(x) = P(\chi^2(f) \leq x)$, where f is a parameter.

f	0.0005	0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.30	0.40	0.50
1	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.06	0.15	0.27	0.45
2	0.00	0.00	0.01	0.02	0.05	0.10	0.21	0.45	0.71	1.02	1.39
3	0.02	0.02	0.07	0.11	0.22	0.35	0.58	1.01	1.42	1.87	2.37
4	0.06	0.09	0.21	0.30	0.48	0.71	1.06	1.65	2.19	2.75	3.36
5	0.16	0.21	0.41	0.55	0.83	1.15	1.61	2.34	3.00	3.66	4.35
6	0.30	0.38	0.68	0.87	1.24	1.64	2.20	3.07	3.83	4.57	5.35
7	0.48	0.60	0.99	1.24	1.69	2.17	2.73	3.82	4.67	5.49	6.35
8	0.71	0.86	1.34	1.65	2.18	2.73	3.49	4.59	5.53	6.42	7.34
9	0.97	1.15	1.73	2.09	2.70	3.33	4.17	5.38	6.39	7.36	8.34
10	1.26	1.48	2.16	2.56	3.25	3.94	4.87	6.18	7.27	8.30	9.34
11	1.59	1.83	2.60	3.05	3.82	4.57	5.58	6.99	8.15	9.24	10.34
12	1.93	2.21	3.07	3.57	4.40	5.23	6.30	7.81	9.03	10.18	11.34
13	2.31	2.62	3.57	4.11	5.01	5.89	7.04	8.63	9.93	11.13	12.34
14	2.70	3.04	4.07	4.66	5.63	6.57	7.79	9.47	10.82	12.08	13.34
15	3.11	3.48	4.60	5.23	6.26	7.26	8.55	10.31	11.72	13.03	14.34
16	3.54	3.94	5.14	5.81	6.91	7.96	9.31	11.15	12.62	13.98	15.34
17	3.98	4.42	5.70	6.41	7.56	8.67	10.09	12.00	13.53	14.94	16.34
18	4.44	4.90	6.26	7.01	8.23	9.39	10.86	12.86	14.44	15.89	17.34
19	4.91	5.41	6.84	7.63	8.91	10.12	11.65	13.72	15.35	16.85	18.34
20	5.40	5.92	7.43	8.26	9.59	10.85	12.44	14.58	16.27	17.81	19.34
21	5.90	6.45	8.03	8.90	10.28	11.59	13.24	15.44	17.18	18.77	20.34
22	6.40	6.98	8.64	9.54	10.98	12.34	14.04	16.31	18.10	19.73	21.34
23	6.92	7.53	9.26	10.20	11.69	13.09	14.85	17.19	19.02	20.69	22.34
24	7.45	8.08	9.89	10.86	12.40	13.85	15.66	18.06	19.94	21.65	23.34
25	7.99	8.65	10.52	11.52	13.12	14.61	16.47	18.94	20.87	22.62	24.34
26	8.54	9.22	11.16	12.20	13.84	15.38	17.29	19.82	21.79	23.58	25.34
27	9.09	9.80	11.81	12.88	14.57	16.15	18.11	20.70	22.72	24.54	26.34
28	9.66	10.39	12.46	13.56	15.31	16.93	18.94	21.59	23.65	25.51	27.34
29	10.23	10.99	13.12	14.26	16.05	17.71	19.77	22.48	24.58	26.48	28.34
30	10.80	11.59	13.79	14.95	16.79	18.49	20.60	23.36	25.51	27.44	29.34
40	16.91	17.92	20.71	22.16	24.43	26.51	29.05	32.34	34.87	37.13	39.34
50	23.46	24.67	27.99	29.71	32.36	34.76	37.69	41.45	46.86	49.33	51.33
60	30.34	31.74	35.53	37.48	40.48	43.19	46.46	50.64	53.81	56.62	59.33
100	59.90	61.92	67.33	70.06	74.22	77.93	82.36	87.95	92.13	95.81	99.33

Table for $\chi^2(f)$ random variable $F(x) = P(\chi^2(f) \leq x)$, where f is a parameter.

f	0.60	0.70	0.80	0.90	0.95	0.975	0.99	0.995	0.999	0.9995
1	0.71	1.07	1.64	2.71	3.84	5.02	6.63	7.88	10.83	12.12
2	1.83	2.41	3.22	4.61	5.99	7.38	9.21	10.60	13.82	15.20
3	2.95	3.66	4.64	6.25	7.81	9.35	11.34	12.84	16.27	17.73
4	4.04	4.88	5.99	7.78	9.49	11.14	13.28	14.86	18.47	20.00
5	5.13	6.06	7.29	9.24	11.07	12.83	15.09	16.75	20.52	22.11
6	6.21	7.23	8.56	10.64	12.59	14.45	16.81	18.55	22.46	24.10
7	7.28	8.38	9.80	12.02	14.07	16.01	18.48	20.28	24.32	26.02
8	8.35	9.52	11.03	13.36	15.51	17.53	20.09	21.95	26.12	27.87
9	9.41	10.66	12.24	14.68	16.92	19.02	21.67	23.59	27.88	29.67
10	10.47	11.78	13.44	15.99	18.31	20.48	23.21	25.19	29.59	31.42
11	11.53	12.90	14.63	17.28	19.68	21.92	24.72	26.76	31.26	33.14
12	12.58	14.01	15.81	18.55	21.03	23.34	26.22	28.30	32.91	34.82
13	13.64	15.12	16.98	19.81	22.36	24.74	27.69	29.82	34.53	36.48
14	14.69	16.22	18.15	21.06	23.68	26.12	29.14	31.32	36.12	38.11
15	15.73	17.32	19.31	22.31	25.00	27.49	30.58	32.80	37.70	39.72
16	16.78	18.42	20.47	23.54	26.30	28.85	32.00	34.27	39.25	41.31
17	17.82	19.51	21.61	24.77	27.59	30.19	33.41	35.72	40.79	42.88
18	18.87	20.60	22.76	25.99	28.87	31.53	34.81	37.16	42.31	44.43
19	19.91	21.69	23.90	27.20	30.14	32.85	36.19	38.58	43.82	45.97
20	20.95	22.77	25.04	28.41	31.41	34.17	37.57	40.00	45.31	47.50
21	21.99	23.86	26.17	29.62	32.67	35.48	38.93	41.40	46.80	49.01
22	23.03	24.94	27.30	30.81	33.92	36.78	40.29	42.80	48.27	50.51
23	24.07	26.02	28.43	32.01	35.17	38.08	41.64	44.18	49.73	52.00
24	25.11	27.10	29.55	33.20	36.42	39.36	42.98	45.56	51.18	53.48
25	26.14	28.17	30.68	34.38	37.65	40.65	44.31	46.93	52.62	54.95
26	27.18	29.25	31.79	35.56	38.89	41.92	45.64	48.29	54.05	56.41
27	28.21	30.32	32.91	36.74	40.11	43.19	46.96	49.64	55.48	57.86
28	29.25	31.39	34.03	37.92	41.34	44.46	48.28	50.99	56.89	59.30
29	30.28	32.46	35.14	39.09	42.56	45.72	49.59	52.34	58.30	60.73
30	31.32	33.53	36.25	40.26	43.77	46.98	50.89	53.67	59.70	62.16
40	41.62	44.16	47.27	51.81	55.76	59.34	63.69	66.77	73.40	76.09
50	51.89	54.72	58.16	63.17	67.50	71.42	76.15	79.49	86.66	89.56
60	62.13	65.23	68.97	74.40	79.08	83.30	88.38	91.95	99.61	102.69
100	102.95	106.91	111.67	118.50	124.34	129.56	135.81	140.17	149.45	153.17

(9.4) Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
 If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1 - p) \geq n - k)$.

n	k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
2	0	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225	0.3600	0.3025	0.2500
	1	0.9975	0.9900	0.9775	0.9600	0.9375	0.9100	0.8775	0.8400	0.7975	0.7500
3	0	0.8574	0.7290	0.6141	0.5120	0.4219	0.3430	0.2746	0.2160	0.1664	0.1250
	1	0.9928	0.9720	0.9392	0.8960	0.8438	0.7840	0.7183	0.6480	0.5747	0.5000
4	0	0.8145	0.6561	0.5220	0.4096	0.3164	0.2401	0.1785	0.1256	0.0915	0.0625
	1	0.9860	0.9477	0.8905	0.8192	0.7383	0.6517	0.5630	0.4735	0.3910	0.3125
5	0	0.7738	0.5905	0.4437	0.3277	0.2373	0.1681	0.1160	0.0778	0.0503	0.0313
	1	0.9774	0.9185	0.8352	0.7373	0.6328	0.5282	0.4284	0.3370	0.2562	0.1875
6	0	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156
	1	0.9672	0.8847	0.7765	0.6554	0.5339	0.4202	0.3191	0.2333	0.1636	0.1094
7	0	0.6983	0.4783	0.3206	0.2097	0.1335	0.0824	0.0490	0.0280	0.0152	0.0078
	1	0.9566	0.8503	0.7166	0.5767	0.4449	0.3294	0.2338	0.1586	0.1024	0.0625
8	0	0.6634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039
	1	0.9428	0.8131	0.6572	0.5033	0.3671	0.2553	0.1691	0.1064	0.0632	0.0352
9	0	0.6302	0.3874	0.2316	0.1342	0.0751	0.0404	0.0207	0.0101	0.0046	0.0020
	1	0.9288	0.7748	0.5995	0.4362	0.3003	0.1960	0.1211	0.0705	0.0385	0.0195

Table for Binomial random variable $P(Bin(n, p) \leq k)$ if $p \leq 0.5$.
 If $p > 0.5$, then $P(Bin(n, p) \leq k) = P(Bin(n, 1 - p) \geq n - k)$.

n	k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
10	0	0.5887	0.3487	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010
	1	0.9139	0.7361	0.5443	0.3758	0.2440	0.1493	0.0860	0.0464	0.0233	0.0107
11	0	0.5688	0.3138	0.1673	0.0859	0.0422	0.0198	0.0088	0.0036	0.0014	0.0005
	1	0.8981	0.6974	0.4922	0.3221	0.1971	0.1130	0.0606	0.0302	0.0139	0.0059
12	0	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057	0.0022	0.0008	0.0002
	1	0.8816	0.6590	0.4435	0.2749	0.1584	0.0850	0.0424	0.0196	0.0083	0.0032

Table for Binomial random variable $P(\text{Bin}(n, p) \leq k)$ if $p \leq 0.5$. If $p > 0.5$, then $P(\text{Bin}(n, p) \leq k) = P(\text{Bin}(n, 1 - p) \geq n - k)$.

Table with columns n, k, p (0.05 to 0.50) and rows for n=14, 15, 16. Each cell contains a probability value.

Table for Binomial random variable $P(\text{Bin}(n, p) \leq k)$ if $p \leq 0.5$. If $p > 0.5$, then $P(\text{Bin}(n, p) \leq k) = P(\text{Bin}(n, 1 - p) \geq n - k)$.

Table with columns n, k, p (0.05 to 0.50) and rows for n=17, 18, 19. Each cell contains a probability value.

(9.5) Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	μ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725	0.7358
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371	0.9197
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865	0.9810
4	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977	0.9963
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9994
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	μ									
	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019	0.1827	0.1653	0.1496	0.1353
1	0.6990	0.6626	0.6268	0.5918	0.5578	0.5249	0.4932	0.4628	0.4337	0.4060
2	0.9004	0.8795	0.8571	0.8335	0.8088	0.7834	0.7572	0.7306	0.7037	0.6767
3	0.9743	0.9662	0.9569	0.9463	0.9344	0.9212	0.9068	0.8913	0.8747	0.8571
4	0.9946	0.9923	0.9893	0.9857	0.9814	0.9763	0.9704	0.9636	0.9559	0.9473
5	0.9990	0.9985	0.9978	0.9968	0.9955	0.9940	0.9920	0.9896	0.9868	0.9834
6	0.9999	0.9997	0.9996	0.9994	0.9991	0.9987	0.9981	0.9974	0.9966	0.9955
7	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9996	0.9994	0.9992	0.9989
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
k	μ									
	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	0.1225	0.1108	0.1003	0.0907	0.0821	0.0743	0.0672	0.0608	0.0550	0.0498
1	0.3796	0.3546	0.3309	0.3084	0.2873	0.2674	0.2487	0.2311	0.2146	0.1991
2	0.6496	0.6227	0.5960	0.5697	0.5438	0.5184	0.4936	0.4695	0.4460	0.4232
3	0.8386	0.8194	0.7993	0.7787	0.7576	0.7360	0.7141	0.6919	0.6696	0.6472
4	0.9379	0.9275	0.9162	0.9041	0.8912	0.8774	0.8629	0.8477	0.8318	0.8153
5	0.9796	0.9751	0.9700	0.9643	0.9580	0.9510	0.9433	0.9349	0.9258	0.9161
6	0.9941	0.9925	0.9906	0.9884	0.9858	0.9828	0.9794	0.9756	0.9713	0.9665
7	0.9985	0.9980	0.9974	0.9967	0.9958	0.9947	0.9934	0.9919	0.9901	0.9881
8	0.9997	0.9995	0.9994	0.9991	0.9989	0.9985	0.9981	0.9976	0.9969	0.9962
9	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996	0.9995	0.9993	0.9991	0.9989
10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	μ									
	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
0	0.0408	0.0334	0.0273	0.0224	0.0183	0.0150	0.0123	0.0101	0.0082	0.0067
1	0.1712	0.1468	0.1257	0.1074	0.0916	0.0780	0.0663	0.0563	0.0477	0.0404
2	0.3799	0.3397	0.3027	0.2689	0.2381	0.2127	0.1851	0.1626	0.1425	0.1247
3	0.6025	0.5584	0.5152	0.4735	0.4335	0.3954	0.3594	0.3257	0.2942	0.2650
4	0.7806	0.7442	0.7064	0.6678	0.6288	0.5898	0.5512	0.5132	0.4763	0.4405
5	0.8946	0.8705	0.8441	0.8156	0.7851	0.7531	0.7199	0.6858	0.6510	0.6160
6	0.9534	0.9421	0.9267	0.9091	0.8893	0.8675	0.8436	0.8180	0.7908	0.7622
7	0.9832	0.9769	0.9682	0.9599	0.9489	0.9361	0.9214	0.9049	0.8867	0.8666
8	0.9943	0.9917	0.9883	0.9840	0.9786	0.9721	0.9642	0.9549	0.9442	0.9319
9	0.9982	0.9973	0.9960	0.9942	0.9919	0.9889	0.9851	0.9805	0.9749	0.9682
10	0.9995	0.9992	0.9987	0.9981	0.9972	0.9959	0.9943	0.9922	0.9896	0.9863
11	0.9999	0.9998	0.9996	0.9994	0.9991	0.9986	0.9980	0.9971	0.9960	0.9945
12	1.0000	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996	0.9993	0.9990	0.9986
13	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9996	0.9993	0.9990	0.9986
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9995	0.9993
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998
k	μ									
	5.2	5.4	5.6	5.8	6.0	6.5	7.0	7.5	8.0	8.5
0	0.0055	0.0045	0.0037	0.0030	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002
1	0.0342	0.0289	0.0244	0.0206	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019
2	0.1088	0.0948	0.0824	0.0715	0.0620	0.0430	0.0286	0.0203	0.0138	0.0093
3	0.2381	0.2133	0.1906	0.1700	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301
4	0.4061	0.3733	0.3422	0.3127	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744
5	0.5809	0.5461	0.5119	0.4783	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496
6	0.7324	0.7017	0.6703	0.6384	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562
7	0.8449	0.8217	0.7970	0.7710	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856
8	0.9181	0.9027	0.8857	0.8672	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231
9	0.9603	0.9512	0.9409	0.9292	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530
10	0.9823	0.9775	0.9718	0.9651	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634
11	0.9927	0.9904	0.9875	0.9841	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487
12	0.9972	0.9962	0.9949	0.9932	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091
13	0.9990	0.9986	0.9980	0.9973	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486
14	0.9999	0.9995	0.9993	0.9990	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726
15	0.9999	0.9998	0.9998	0.9996	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862
16	1.0000	0.9999	0.9999	0.9999	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934
17	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970
18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9993	0.9987
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9995
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table for Poisson random variable $P(Po(\mu) \leq k)$.

k	μ														
	9.0	9.5	10.0	11.0	12.0	13.0	14.0	15.0	16.0	17.0					
0	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000					
1	0.0012	0.0008	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000					
2	0.0062	0.0042	0.0028	0.0012	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000					
3	0.0212	0.0149	0.0103	0.0049	0.0023	0.0011	0.0005	0.0002	0.0001	0.0000					
4	0.0550	0.0403	0.0293	0.0151	0.0076	0.0037	0.0018	0.0009	0.0004	0.0002					
5	0.1157	0.0885	0.0671	0.0375	0.0203	0.0107	0.0055	0.0028	0.0014	0.0007					
6	0.2068	0.1649	0.1301	0.0786	0.0458	0.0259	0.0142	0.0076	0.0040	0.0021					
7	0.3239	0.2687	0.2202	0.1432	0.0895	0.0540	0.0316	0.0180	0.0100	0.0054					
8	0.4557	0.3918	0.3328	0.2320	0.1550	0.0998	0.0621	0.0374	0.0220	0.0126					
9	0.5874	0.5218	0.4579	0.3405	0.2424	0.1658	0.1094	0.0699	0.0433	0.0261					
10	0.7060	0.6453	0.5830	0.4599	0.3472	0.2517	0.1757	0.1185	0.0774	0.0491					
11	0.8030	0.7520	0.6968	0.5793	0.4616	0.3532	0.2600	0.1848	0.1270	0.0847					
12	0.8758	0.8364	0.7916	0.6887	0.5760	0.4631	0.3585	0.2676	0.1931	0.1350					
13	0.9261	0.8981	0.8645	0.7813	0.6815	0.5730	0.4644	0.3632	0.2745	0.2009					
14	0.9585	0.9400	0.9165	0.8540	0.7720	0.6751	0.5704	0.4657	0.3675	0.2808					
15	0.9780	0.9665	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681	0.4667	0.3715					
16	0.9889	0.9823	0.9730	0.9441	0.8987	0.8355	0.7559	0.6641	0.5660	0.4677					
17	0.9947	0.9911	0.9857	0.9678	0.9370	0.8905	0.8272	0.7489	0.6593	0.5640					
18	0.9976	0.9957	0.9928	0.9823	0.9626	0.9302	0.8826	0.8195	0.7423	0.6550					
19	0.9989	0.9980	0.9965	0.9907	0.9787	0.9573	0.9235	0.8752	0.8122	0.7363					
20	0.9996	0.9991	0.9984	0.9953	0.9884	0.9750	0.9521	0.9170	0.8682	0.8055					
21	0.9998	0.9996	0.9993	0.9977	0.9939	0.9859	0.9712	0.9469	0.9108	0.8615					
22	0.9999	0.9999	0.9997	0.9990	0.9970	0.9924	0.9833	0.9673	0.9418	0.9047					
23	1.0000	0.9999	0.9999	0.9995	0.9985	0.9960	0.9907	0.9805	0.9633	0.9367					
24	1.0000	1.0000	1.0000	0.9998	0.9993	0.9980	0.9950	0.9888	0.9777	0.9594					
25	1.0000	1.0000	1.0000	0.9999	0.9997	0.9990	0.9974	0.9938	0.9869	0.9748					
26	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9987	0.9967	0.9925	0.9848					
27	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9983	0.9959	0.9912					
28	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9978	0.9950					
29	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9994	0.9986					
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993					
31	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9996					
32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9996					
33	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999					
34	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999					
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000					