1. $\left.\begin{array}{l} U=\frac{X}{X+Y} \\ V=X+Y \end{array}\right\} \Rightarrow \begin{array}{l} X=UV \\ Y=V-UV \end{array}$ $\left(\xrightarrow{X,Y\ge 0} \begin{array}{l} 0<u<1 \\ v>0 \end{array} \right)$ and $J=v \Rightarrow f_{U,V}(u,v)=f_{X,Y}(uv, v-uv)\cdot|J| = v\cdot e^{-v}$, $v>0$, $0<u<1$

$\Rightarrow f_U(u)=\int_0^\infty f_{U,V}(u,v)\,dv = 1$, $0<u<1$ $\qquad f_{U,V}(u,v)=f_U(u)\cdot f_V(v)$

$f_V(v)=\int_0^1 f_{U,V}(u,v)\,du = v\cdot e^{-v}$, $v>0$ $\qquad \Rightarrow U, V$ are independent!

TAMS46: Probability Theory (Second Course) | Provkod: TEN1 | 30 October 2020, 08:00-12:00

Examiner: Xiangfeng Yang (013-285788). **Things allowed:** a calculator, an English-Swedish dictionary.
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.

1 (3 points)

Let X and Y be independent $Exp(1)$-distributed random variables. Show that $X/(X+Y)$ and $X+Y$ are independent, and find their density functions $f_{X/(X+Y)}(u)$ and $f_{X+Y}(v)$.

2. $E(Y)=E(E(Y|X))=E(nX)=nE(X)=n/2$

$Cov(X,Y)=E(XY)-E(X)E(Y)=E(E(XY|X))-\frac{1}{2}\cdot\frac{n}{2}=E(X\,E(Y|X))-\frac{n}{4}$

$=E(X\cdot nX)-\frac{n}{4}=nE(X^2)-\frac{n}{4}=\frac{n}{3}-\frac{n}{4}=\frac{n}{12}$

2 (3 points)

Let Y be a Binomial random variable with a random parameter X as follows:

$$Y\,|\,X=x \sim Bin(n,x), \quad \text{with } X\sim U(0,1). \quad [\text{This can be also written as } Y\,|\,X\sim Bin(n,X)]$$

Compute the expectation $E(Y)$ of Y and the covariance $Cov(X,Y)$ of X and Y.

3. Characteristic function $\left.\begin{array}{l}\varphi_X(t)=(q+pe^{it})^n \\ \varphi_Y(t)=(q+pe^{it})^m\end{array}\right\} \Rightarrow \varphi_{X+Y}(t)=Ee^{it(X+Y)}=\varphi_X(t)\cdot\varphi_Y(t)=(q+pe^{it})^{m+n}$

$\Rightarrow X+Y\sim Bin(n+m,p)$

3 (3 points)

Prove, with the aid of suitable transforms, that if $X\sim Bin(n,p)$ and $Y\sim Bin(m,p)$ are independent, then $X+Y\sim Bin(n+m,p)$.

4 (3 points)

The random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ are independent $U(0,1)$-distributed.
(4.1) Find the density function $f_{X_{(n)}}(x)$ of $X_{(n)}$, where $X_{(n)}=\max\{X_1,X_2,\ldots,X_n\}$.
(4.2) Find the probability $P\left(\frac{\max\{X_{(n)},Y_{(n)}\}}{\min\{X_{(n)},Y_{(n)}\}}\ge 2\right)$.

4. (4.1) It is from P102 book, $f_{X_{(n)}}(x)=n(F(x))^{n-1}f(x)=n\cdot x^{n-1}$, $0<x<1$

(4.2) $P\left(\frac{\max\{X_{(n)},Y_{(n)}\}}{\min\{X_{(n)},Y_{(n)}\}}\ge 2\right)=P(\max\{X_{(n)},Y_{(n)}\}\ge 2\min\{X_{(n)},Y_{(n)}\})$

$=P(X_{(n)}\ge 2Y_{(n)})+P(Y_{(n)}\ge 2X_{(n)})=2^{-n}$.

Symmetry $\Rightarrow P(X_{(n)}\ge 2Y_{(n)})=P(Y_{(n)}\ge 2X_{(n)})$ where $P(X_{(n)}\ge 2Y_{(n)})=\iint_{\{x\ge 2y\}} f_{X_{(n)},Y_{(n)}}(x,y)\,dxdy=\iint_{\substack{\{x\ge 2y\}\\0<x<1\\0<y<1}} nx^{n-1}\cdot ny^{n-1}\,dxdy = 2^{-n}/2$

5 (3 points)

Let $\mathbf{X}=\begin{pmatrix}X_1\\X_2\end{pmatrix}\sim N(\mathbf{0},\Lambda)$ with $\Lambda=\begin{pmatrix}3&1\\1&2\end{pmatrix}$, that is, the density function is $f_{\mathbf{X}}(\mathbf{x})=\frac{1}{2\pi\cdot\sqrt{\det(\Lambda)}}\exp\{-\frac{1}{2}\mathbf{x}'\Lambda^{-1}\mathbf{x}\}$.
Find the conditional density function $f_{X_1+X_2|X_1-X_2=0}(x)$ of X_1+X_2 given that $X_1-X_2=0$.
(Hints: you might need to use the inverse formula $\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$.)

5. $Y=\begin{pmatrix}X_1+X_2\\X_1-X_2\end{pmatrix}=BX$, $B=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$ $\Rightarrow Y\sim N\left(0,\begin{pmatrix}7&1\\1&3\end{pmatrix}\right)\Rightarrow f_Y(y_1,y_2)=\frac{1}{2\pi\sqrt{20}}\exp\{-\frac{1}{40}(3y_1^2-2y_1y_2+7y_2^2)\}$

$Y_2\sim N(0,3)\Rightarrow f_{Y_2}(y_2)=\frac{1}{\sqrt{2\pi}\sqrt{3}}\exp\{-\frac{1}{2}y_2^2/3\}$

$\Rightarrow f_{Y_1|Y_2=0}(y_1)=\frac{f_Y(y_1,0)}{f_{Y_2}(0)}=\frac{\frac{1}{2\pi\sqrt{20}}\exp\{-\frac{1}{40}(3y_1^2)\}}{\frac{1}{\sqrt{2\pi}\sqrt{3}}}=\frac{1}{\sqrt{2\pi}\sqrt{20/3}}\exp\{-\frac{3y_1^2}{40}\}$ $\quad N(0,\frac{20}{3})$

6 (3 points)

(6.1) Let $P(Z_n=0)=1-\frac{1}{n}, P(Z_n=1)=\frac{1}{2n}$ and $P(Z_n=-1)=\frac{1}{2n}$ for $n\ge 1$. Prove that $Z_n \xrightarrow{p} 0$.
(6.2) Let Y be a Normal random variable with a random parameter X as follows:

$$Y\,|\,X=x\sim N(0,x), \quad \text{with } X\sim Po(\lambda). \quad [\text{This can be also written as } Y\,|\,X\sim N(0,X)]$$

Prove that $Y/\sqrt{\lambda}\xrightarrow{d} N(0,1)$ as $\lambda\to\infty$.
(Hint: use the characteristic function of $Y/\sqrt{\lambda}$, and a fact $e^x=1+x+o(x)$ as $x\to 0$.)

6. (6.1) $\forall\varepsilon>0$, $P(|Z_n-0|>\varepsilon)=P(Z_n>\varepsilon)+P(Z_n<-\varepsilon)=\begin{cases}\frac{1}{n} & (\text{if } \varepsilon<1)\\ 0 & (\text{if } \varepsilon\ge 1)\end{cases}\to 0$ as $n\to\infty$

(6.2) characteristic $\varphi_Y(t)=Ee^{itY}=E(E(e^{itY}|X))\overset{\text{P384 book}}{=}E(e^{-\frac{1}{2}t^2X})\overset{\text{P236 book}}{=}e^{\lambda(e^{-\frac{t^2}{2}}-1)}$

$\Rightarrow \varphi_{Y/\sqrt{\lambda}}(t)=Ee^{itY/\sqrt{\lambda}}=\varphi_Y(t/\sqrt{\lambda})=e^{\lambda(e^{-\frac{t^2}{2\lambda}}-1)}\overset{e^x=1+x+o(x)}{=}e^{\lambda(-\frac{t^2}{2\lambda}+o(\frac{1}{\lambda}))}=e^{-\frac{t^2}{2}+o(1)}\xrightarrow{\lambda\to\infty}e^{-\frac{t^2}{2}}=\varphi_{N(0,1)}(t)$

$\Rightarrow Y/\sqrt{\lambda}\xrightarrow{d}N(0,1)$ as $\lambda\to\infty$