Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, an English-Swedish dictionary.
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.

1 (3 points)

Suppose that X and Y have a joint probability density function as follows

$$f(x,y)=\begin{cases}\frac{1}{y}\cdot e^{-x/y}\cdot e^{-y}, & \text{for } 0<x<\infty \text{ and } 0<y<\infty,\\ 0, & \text{otherwise.}\end{cases}$$

Are X/Y and Y independent? Why?

Solution. Let $U=X/Y$ and $V=Y$, then it follows that $X=UV$ and $Y=V$ which gives $J=v$. Thus the joint probability density function of (U,V) is: for $0<u,v<\infty$,

$$f_{U,V}(u,v)=f(uv,v)\cdot|J|=\frac{1}{v}\cdot e^{-(uv)/v}\cdot e^{-v}\cdot v=e^{-u}\cdot e^{-v}.$$

The marginal probability density functions are

$$f_U(u)=\int_{-\infty}^{\infty}f_{U,V}(u,v)dv=\int_0^{\infty}e^{-u}\cdot e^{-v}dv=e^{-u},\quad 0<u<\infty,$$

$$f_V(v)=\int_{-\infty}^{\infty}f_{U,V}(u,v)du=\int_0^{\infty}e^{-u}\cdot e^{-v}du=e^{-v},\quad 0<v<\infty.$$

It is clear that $f_{U,V}(u,v)=f_U(u)\cdot f_V(v)$, so U and V (namely X/Y and Y) are independent. □

2 (3 points)

Let X be a Binomial random variable with a random parameter N as follows:

$$X|N=n\sim Bin(n,p),\qquad \text{with } N\sim Po(\lambda).$$

Find the probability $P(X=k)$ for $k=0,1,2,\ldots$.

Solution. It is from total probability that

$$\begin{aligned}P(X=k)&=\sum_{n=0}^{\infty}P(X=k|N=n)\cdot P(N=n)\\&=\sum_{n=k}^{\infty}\binom{n}{k}p^kq^{n-k}\cdot e^{-\lambda}\frac{\lambda^n}{n!}\\&=\frac{p^k}{k!}e^{-\lambda}\sum_{n=k}^{\infty}\frac{\lambda^n}{(n-k)!}q^{n-k}=\frac{(\lambda p)^k}{k!}e^{-\lambda}\sum_{n=k}^{\infty}\frac{(\lambda q)^{n-k}}{(n-k)!}\\&=\frac{(\lambda p)^k}{k!}e^{-\lambda}\sum_{j=0}^{\infty}\frac{(\lambda q)^j}{j!}=\frac{(\lambda p)^k}{k!}e^{-\lambda}e^{\lambda q}\\&=\frac{(\lambda p)^k}{k!}e^{-\lambda p},\end{aligned}$$

namely $X\sim Po(\lambda p)$. □

3 (3 points)

Let X be a discrete random variable with

$$P(X=0)=0.2, \quad P(X=1)=0.5, \quad P(X=2)=0.3.$$

(3.1) (1p) Find the probability generating function $g_X(t)=E(t^X)$.
(3.2) (1p) Find the moment generating function $\psi_X(t)=E(e^{tX})$.
(3.3) (1p) Find the characteristic generating function $\varphi_X(t)=E(e^{itX})$.

Solution. (3.1)

$$g_X(t)=E(t^X)=t^0\cdot 0.2+t^1\cdot 0.5+t^2\cdot 0.3=0.2+0.5t+0.3t^2, \quad t\in\mathbb{R}.$$

(3.2)

$$\psi_X(t)=E(e^{tX})=e^{t\cdot 0}\cdot 0.2+e^{t\cdot 1}\cdot 0.5+e^{t\cdot 2}\cdot 0.3=0.2+0.5e^t+0.3e^{2t}, \quad t\in\mathbb{R}.$$

(3.3)

$$\varphi_X(t)=E(e^{itX})=e^{it\cdot 0}\cdot 0.2+e^{it\cdot 1}\cdot 0.5+e^{it\cdot 2}\cdot 0.3=0.2+0.5e^{it}+0.3e^{2it}, \quad t\in\mathbb{R}.$$

□

4 (3 points)

Let $X_1\sim Exp(1)$ and $X_2\sim Exp(1)$ be two independent exponential random variables. Set $X_{(1)}=\min\{X_1,X_2\}$ and $X_{(2)}=\max\{X_1,X_2\}$. Find the conditional expectation $E(X_{(2)}|X_{(1)}=x)$.

Solution. Step 1: find the joint probability density function of $(X_{(1)},X_{(2)})$. To this end, for any $0<x<y$,

$$P(X_{(1)}>x,X_{(2)}\le y)=P(x<X_1\le y,x<X_2\le y)=P(x<X_1\le y)\cdot P(x<X_2\le y)=(e^{-x}-e^{-y})^2.$$

Therefore

$$F(x,y):=P(X_{(1)}\le x,X_{(2)}\le y)=P(X_{(2)}\le y)-P(X_{(1)}>x,X_{(2)}\le y)=(1-e^{-y})^2-(e^{-x}-e^{-y})^2.$$

By taking the partial derivatives $\partial^2F(x,y)/\partial x\partial y$, it follows that the joint probability density function is

$$f_{X_{(1)},X_{(2)}}(x,y)=2e^{-x}e^{-y}, \quad 0<x<y.$$

Step 2: find the marginal probability density function of $X_{(1)}$:

$$f_{X_{(1)}}(x)=\int_x^\infty f_{X_{(1)},X_{(2)}}(x,y)dy=2e^{-2x}, 0<x<\infty.$$

Step 3: find the conditional probability density function of $X_{(2)}|X_{(1)}=x$:

$$f_{X_{(2)}|X_{(1)}=x}(y)=\frac{f_{X_{(1)},X_{(2)}}(x,y)}{f_{X_{(1)}}(x)}=\frac{2e^{-x}e^{-y}}{2e^{-2x}}=e^x\cdot e^{-y}, \quad 0<x<y.$$

Step 4:

$$E(X_{(2)}|X_{(1)}=x)=\int_{-\infty}^\infty yf_{X_{(2)}|X_{(1)}=x}(y)dy=\int_x^\infty ye^xe^{-y}dy=\ldots=x+1, \quad x>0.$$

□

5 (3 points)

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean 0 and variance 1. Assume that $N \sim Po(\lambda)$ is independent of $X_1, X_2, \ldots$. Show that

$$\frac{X_1 + X_2 + \ldots + X_N}{\sqrt{N}} \xrightarrow{d} N(0,1) \quad \text{as } \lambda \to \infty.$$

Solution. We first rewrite

$$\frac{X_1 + X_2 + \ldots + X_N}{\sqrt{N}} = \frac{X_1 + X_2 + \ldots + X_N}{\sqrt{\lambda}} \cdot \frac{1}{\sqrt{N/\lambda}}.$$

Now we will show two things: (i) $N/\lambda \xrightarrow{p} 1$. To see this, the moment generating function gives

$$\psi_{N/\lambda}(t) = Ee^{tN/\lambda} = (\text{see Appendex B}) = e^{\lambda(e^{t/\lambda}-1)} = e^{\lambda(t/\lambda + o(1/\lambda))} = e^{t+o(1)} \to e^t = \psi_1(t).$$

(ii) $\frac{X_1+X_2+\ldots+X_N}{\sqrt{\lambda}} \xrightarrow{d} N(0,1)$. To see this, the characteristic function gives

$$\begin{aligned}
\varphi_{\frac{X_1+X_2+\ldots+X_N}{\sqrt{\lambda}}}(t) &= Ee^{it\frac{X_1+X_2+\ldots+X_N}{\sqrt{\lambda}}} = E\left(Ee^{it\frac{X_1+X_2+\ldots+X_N}{\sqrt{\lambda}}}|N\right) \\
&= \sum_{k\geq 0} E(e^{it\frac{X_1+X_2+\ldots+X_k}{\sqrt{\lambda}}}|N=k) \cdot P(N=k) \\
&= \sum_{k\geq 0} Ee^{it\frac{X_1+X_2+\ldots+X_k}{\sqrt{\lambda}}} \cdot P(N=k), \quad (\text{as } N \text{ is independent of } X_1, X_2, \ldots) \\
&= \sum_{k\geq 0} \left(E(e^{i\frac{t}{\sqrt{\lambda}}X_1})\right)^k \cdot P(N=k), \quad (E(e^{i\frac{t}{\sqrt{\lambda}}X_1}) = 1 - \frac{t^2}{2\lambda} + o(1/\lambda) \text{ as mean 0 and variance 1}) \\
&= e^{-\lambda} \sum_{k\geq 0} \left(1 - \frac{t^2}{2\lambda} + o(1/\lambda)\right)^k \cdot \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k\geq 0} \frac{\left(\lambda - t^2/2 + o(1)\right)^k}{k!} = e^{-\lambda} \cdot e^{\lambda - t^2/2 + o(1)} = e^{-t^2/2+o(1)} \to e^{-t^2/2} = \varphi_{N(0,1)}(t).
\end{aligned}$$

Now (i) and (ii) plus the Cramér's theorem (Book Theorem 6.5 on Page 168) imply that

$$\frac{X_1 + X_2 + \ldots + X_N}{\sqrt{N}} \xrightarrow{d} N(0,1) \quad \text{as } \lambda \to \infty.$$

□

6 (3 points)

Let $X_1, X_2, \ldots$ be independent $U(0,1)$ random variables, and set $Y_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n$ for $n = 1, 2, \ldots$.
(6.1) (1p) Prove that $Y_n \xrightarrow{p} 0$ as $n \to \infty$.

(6.2) (2p) Prove that $Y_n \xrightarrow{a.s.} 0$ as $n \to \infty$.

Solution. For any $1 > \epsilon > 0$, it is important to notice that

$$\{X_1 \cdot X_2 \cdot \ldots \cdot X_n > \epsilon\} \subseteq \{X_1 > \epsilon, X_2 > \epsilon, \ldots, X_n > \epsilon\}.$$

There is no need to consider $\epsilon \geq 1$ as in this case everything becomes 0.
(6.1) For any $1 > \epsilon > 0$,

$$\begin{aligned}
P(|Y_n - 0| > \epsilon) = P(Y_n > \epsilon) &= P(X_1 \cdot X_2 \cdot \ldots \cdot X_n > \epsilon) \\
&\leq P(X_1 > \epsilon, X_2 > \epsilon, \ldots, X_n > \epsilon) \\
&= P(X_1 > \epsilon) \cdot P(X_2 > \epsilon) \cdot \ldots \cdot P(X_n > \epsilon) \\
&= (1-\epsilon)^n \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore $Y_n \xrightarrow{p} 0$ as $n \to \infty$.

(6.2) For any $1 > \epsilon > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|Y_n - 0| > \epsilon) = \sum_{n=1}^{\infty} P(Y_n > \epsilon) &= \sum_{n=1}^{\infty} P(X_1 \cdot X_2 \cdot \ldots \cdot X_n > \epsilon) \\
&\leq \sum_{n=1}^{\infty} P(X_1 > \epsilon, X_2 > \epsilon, \ldots, X_n > \epsilon) \\
&= \sum_{n=1}^{\infty} P(X_1 > \epsilon) \cdot P(X_2 > \epsilon) \cdot \ldots \cdot P(X_n > \epsilon) \\
&= \sum_{n=1}^{\infty} (1-\epsilon)^n = \frac{1}{\epsilon} - 1 < \infty.
\end{aligned}$$

Therefore the first Borel-Cantelli lemma (Book section 7.7) implies that $Y_n \xrightarrow{a.s.} 0$ as $n \to \infty$.

$\square$

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n = 1, 2, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$	$\dfrac{p}{1 - qe^{it}}$
First success $\mathrm{Fs}(p)$, $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	$\dfrac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n = 1, 2, 3, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\dfrac{q}{p}$	$n\dfrac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m > 0$	$p(k) = e^{-m}\dfrac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it} - 1)}$
Hypergeometric $H(N,n,p)$, $n = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $p = 0, \dfrac{1}{N}, \dfrac{2}{N}, \ldots, 1$	$p(k) = \dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\ k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\,\dfrac{N - n}{N - 1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\operatorname{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$ $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\operatorname{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\operatorname{Exp}(a),\ a>0$	$f(x)=\frac{1}{a}\,e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\frac{1}{\Gamma(p)}\,x^{p-1}\,\frac{1}{a^p}\,e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\dots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1}$, $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1) -\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2}$, $n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2}$, $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\left(\frac{n}{n-2}\right)^2$, $n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\|t\|}$
$C(0,1)$	$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\|t\|}$
Pareto	$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}},\ x > k$	$\frac{\alpha k}{\alpha - 1},\ \alpha > 1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$				