Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '

1 (3 points)

(1.1) (1p) Let X be a continuous one-dimensional random variable with a probability density function $f_X(x), x \in \mathbb{R}$. Define $Y = X^2$, find the probability density function $f_Y(y)$ of Y.
(1.2) (2p) Let X_1 and X_2 be independent $Exp(1)$-distributed random variables. Find the density function of $\frac{X_1}{X_1+X_2}$.

Solution. (1.1) It is from the many-to-one formula (#2.2 on p.23 book) that

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}.$$

(1.2) Let $U = \frac{X_1}{X_1+X_2}$ and $V = X_1 + X_2$, then it follows that $X_1 = U \cdot V$ and $X_2 = V - U \cdot V$. Furthermore, it is from $x_1 > 0$ and $x_2 > 0$ that

$$0 < u < 1, \qquad v > 0.$$

Therefore

$$f_{U,V}(u, v) = f_{X_1,X_2}(uv, v - uv) \cdot |\mathbf{J}| = e^{-uv} \cdot e^{-(v-uv)} \cdot v = ve^{-v}, \quad 0 < u < 1, v > 0.$$

Then

$$f_U(u) = \int_0^\infty f_{U,V}(u, v)dv = \int_0^\infty ve^{-v}dv = 1, \quad 0 < u < 1.$$

□

2 (3 points)

Let X be a Poisson random variable with a random parameter M as follows:

$$X|M = m \sim Po(m), \qquad \text{with } M \sim Exp(1).$$

(2.1) (1p) Find the mean $E(X)$ of X.
(2.2) (1p) Find $E(X \cdot M)$.
(2.3) (1p) Find the probability $P(X = 1)$.

Solution. (2.1) $E(X) = E(E(X|M)) = E(M) = 1$.
(2.2)

$$E(X \cdot M) = E(E(X \cdot M|M)) = E(ME(X|M)) = E(M^2) = \int_0^\infty x^2e^{-x}dx = 2.$$

(2.3) It is from total probability that

$$P(X = 1) = \int_0^\infty P(X = 1|M = m) \cdot f_M(m)dm = \int_0^\infty e^{-m}m \cdot e^{-m}dm = \int_0^\infty me^{-2m}dm = \frac{1}{4}.$$

□

3 (3 points)

Suppose that X is a random variable such that

$$E(X^n) = \frac{1}{4} + 2^{n-1}, \quad n = 1, 2, \dots.$$

(3.1) (2p) Find the moment generating function $\psi_X(t)$ of X.
(3.2) (1p) Determine the probabilities $P(X = k)$ for $k = 0, 1, 2, \dots$.

Solution. (3.1) The moment generating function is

$$\begin{aligned}
\psi_X(t) = E(e^{tX}) &= \sum_{n=0}^{\infty} \frac{t^n E(X^n)}{n!} = 1 + \sum_{n=1}^{\infty} \frac{t^n E(X^n)}{n!} = 1 + \sum_{n=1}^{\infty} \frac{t^n(\frac{1}{4} + 2^{n-1})}{n!} \\
&= 1 + \frac{1}{4}\sum_{n=1}^{\infty} \frac{t^n}{n!} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2t)^n}{n!} = 1 + \frac{1}{4}(e^t - 1) + \frac{1}{2}(e^{2t} - 1) \\
&= \frac{1}{4} + \frac{1}{4}e^t + \frac{1}{2}e^{2t}.
\end{aligned}$$

(3.2) If one considers a random variable Y as: $P(Y = 0) = \frac{1}{4}, P(Y = 1) = \frac{1}{4}$ and $P(Y = 2) = \frac{1}{2}$, then the moment generating function of Y is

$$\psi_Y(t) = E(e^{tY}) = \frac{1}{4} + \frac{1}{4}e^t + \frac{1}{2}e^{2t}.$$

The fact $\psi_X(t) = \psi_Y(t)$ implies that X and Y have the same distribution, namely $P(X = 0) = \frac{1}{4}, P(X = 1) = \frac{1}{4}$ and $P(X = 2) = \frac{1}{2}$.

□

4 (3 points)

Suppose that X_1, X_2 and X_3 are independent $U(0, 1)$ random variables, and let $(X_{(1)}, X_{(2)}, X_{(3)})$ be the corresponding order statistic.
(4.1) (2p) Find the conditional probability density function $f_{X_{(3)}|X_{(1)}=y_1}(y_3)$ of $X_{(3)}$ given $X_{(1)} = y_1$.
(4.2) (1p) Find the probability $P(X_{(3)} \geq 2X_{(1)})$.

Solution. (4.1) It is from Theorem 3.1 (p.110 book) that the joint probability density function of $(X_{(1)}, X_{(2)}, X_{(3)})$ is

$$f_{X_{(1)},X_{(2)},X_{(3)}}(y_1, y_2, y_3) = 6, \qquad 0 < y_1 < y_2 < y_3 < 1.$$

Therefore, the joint probability density function of $(X_{(1)}, X_{(3)})$ is

$$f_{X_{(1)},X_{(3)}}(y_1, y_3) = \int_{y_1}^{y_3} f_{X_{(1)},X_{(2)},X_{(3)}}(y_1, y_2, y_3)dy_2 = 6(y_3 - y_1), \qquad 0 < y_1 < y_3 < 1.$$

This further implies that the probability density function of $X_{(1)}$ is

$$f_{X_{(1)}}(y_1) = \int_{y_1}^{1} f_{X_{(1)},X_{(3)}}(y_1, y_3)dy_3 = 3(1 - y_1)^2, \quad 0 < y_1 < 1.$$

Thereofore

$$f_{X_{(3)}|X_{(1)}=y_1}(y_3) = \frac{f_{X_{(1)},X_{(3)}}(y_1, y_3)}{f_{X_{(1)}}(y_1)} = \frac{2(y_3 - y_1)}{(1 - y_1)^2}, \quad 0 < y_1 < y_3 < 1.$$

(4.2)

$$P(X_{(3)} \geq 2X_{(1)}) = \int_0^{1/2} \left(\int_{2y_1}^{1} f_{X_{(1)},X_{(3)}}(y_1, y_3)dy_3 \right) dy_1 = 6\int_0^{1/2} (1/2 - y_1)dy_1 = 3/4 = 0.75.$$

□

5 (3 points)

Let $\mathbf{X} = (X_1, X_2)'$ be a two dimensional normal random variable $\mathbf{X} \sim N(\mu, \Lambda)$, where the mean vector is $\mu = (0,0)'$ and the covariance matrix is $= \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Define a new two dimensional random variable as $\mathbf{Y} = (Y_1, Y_2)'$ with $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

(5.1) (1.5p) Find the distribution of $\mathbf{Y}$.

(5.2) (1.5p) Find the conditional distribution of Y_2 given $Y_1 = 1$.

Solution. (5.1) $\mathbf{Y}$ can be written as $\mathbf{Y} = \mathbf{B}\mathbf{X}$ with $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Therefore the distribution of $\mathbf{Y}$ is $\mathbf{Y} \sim N(\mathbf{B}\mu, \mathbf{B}\Lambda\mathbf{B}') = N(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 1 & 3 \end{pmatrix})$.

(5.2) According to #(6.2) (p.127, book), the conditional distribution Y_2 given $Y_1 = 1$ is still normal with

$$E(Y_2|Y_1 = 1) = \rho\frac{\sigma_{Y_2}}{\sigma_{Y_1}} = \frac{1}{7}, \text{ and } Var(Y_2|Y_1 = 1) = \sigma_{Y_2}^2(1 - \rho^2) = \frac{20}{7},$$

where $\sigma_{Y_1}^2 = 7, \sigma_{Y_2}^2 = 3$ and $1 = cov(Y_1, Y_2) = \rho\sigma_{Y_1}\sigma_{Y_2}$, namely $\rho = 1/\sqrt{21}$. That is

$$Y_2|Y_1 = 1 \sim N(\frac{1}{7}, \frac{20}{7}).$$

□

6 (3 points)

Let $X_1, X_2, \ldots$ be i.i.d. (independent and identically distributed) random variables with finite mean $\mu = E(X_i) \neq 0$ and finite variance $\sigma^2 = Var(X_i) \neq 0$. Let $S_n = X_1 + X_2 + \ldots + X_n$ for $n \geq 1$.

(6.1) (1p)

Does $\dfrac{S_n - n\mu}{S_n + n\mu}$ converge in probability? If yes, then find the limit; if no, then explain why.

(6.2) (2p)

Does $\sqrt{n} \cdot \dfrac{S_n - n\mu}{S_n + n\mu}$ converge in distribution? If yes, then find the limit; if no, then explain why.

Solution. (6.1) Yes! It is from LLN (p.162, book) that $\frac{S_n}{n} \xrightarrow{p} \mu$, therefore Cramér's theorem (p.168,book) implies

$$\frac{S_n - n\mu}{S_n + n\mu} = \frac{\frac{S_n}{n} - \mu}{\frac{S_n}{n} + \mu} \xrightarrow{p} \frac{\mu - \mu}{\mu + \mu} = 0.$$

(6.2) Yes! It is from LLN (p.162, book) that $\frac{S_n}{n} \xrightarrow{p} \mu$, and from CLT (p.162, book) that $\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$, therefore Cramér's theorem (p.168,book) implies

$$\sqrt{n} \cdot \frac{S_n - n\mu}{S_n + n\mu} = \sigma \cdot \frac{\frac{S_n - n\mu}{\sigma\sqrt{n}}}{\frac{S_n}{n} + \mu} \xrightarrow{d} \sigma \cdot \frac{N(0,1)}{\mu + \mu} = N(0, \frac{\sigma^2}{4\mu^2}).$$

□

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a)=1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1)=p(1)=\frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0\le p\le 1$	$p(0)=q,\ p(1)=p;\ q=1-p$	p	pq	$q+pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n=1,2,\ldots$, $0\le p\le 1$	$p(k)=\binom{n}{k}p^kq^{n-k},\ k=0,1,\ldots,n;\ q=1-p$	np	npq	$(q+pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0\le p\le 1$	$p(k)=pq^k,\ k=0,1,2,\ldots;\ q=1-p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^{it}}$
First success $\mathrm{Fs}(p)$, $0\le p\le 1$	$p(k)=pq^{k-1},\ k=1,2,\ldots;\ q=1-p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1-qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n=1,2,3,\ldots$, $0\le p\le 1$	$p(k)=\binom{n+k-1}{k}p^nq^k,\ k=0,1,2,\ldots;$ $q=1-p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1-qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m>0$	$p(k)=e^{-m}\frac{m^k}{k!},\ k=0,1,2,\ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N,n,p)$, $n=0,1,\ldots,N$, $N=1,2,\ldots$, $p=0,\frac{1}{N},\frac{2}{N},\ldots,1$	$p(k)=\dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k=0,1,\ldots,Np;$ $q=1-p;$ $n-k=0,\ldots,Nq$	np	$npq\,\frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$, $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a>0$	$f(x)=\frac{1}{a}\,e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\frac{1}{\Gamma(p)}\,x^{p-1}\,\frac{1}{a^p}\,e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\ldots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1}$, $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^\beta\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)-\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},\ -\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},\ -\infty<x<\infty$	0	$\dfrac{n}{n-2}$, $n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},\ x>0$	$\frac{n}{n-2}$, $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2$, $n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\lvert t\rvert}$
$C(0,1)$	$f(x) = \dfrac{1}{\pi} \cdot \dfrac{1}{1+x^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\lvert t\rvert}$
Pareto	$f(x) = \dfrac{\alpha k^\alpha}{x^{\alpha+1}},\ x > k$	$\dfrac{\alpha k}{\alpha-1},\ \alpha > 1$	$\dfrac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$				