TAMS46: Probability Theory (Second Course) | Provkod: TEN1 | 05 January 2023, 14:00-18:00

Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '

1 (3 points)

Let $X \sim U(0,1)$ and $Y \sim Exp(1)$ be independent random variables. Find the probability density function of $X+Y$.

Solution. It is clear that $f_X(x) = 1$ for $0 < x < 1$, and $f_Y(y) = e^{-y}$ for $y > 0$. Then it is directly from the convolution formula that

$$\begin{aligned} f_{X+Y}(u) &= \int_{-\infty}^{\infty} f_X(x) f_Y(u-x)dx = \int_0^1 1 \cdot f_Y(u-x)dx \\ &= \begin{cases} \int_0^1 1 \cdot e^{-(u-x)}dx, & \text{if } u \geq 1 \\ \int_0^u 1 \cdot e^{-(u-x)}dx, & \text{if } 0 < u < 1 \end{cases} \\ &= \begin{cases} e^{-u}(e-1), & \text{if } u \geq 1 \\ 1-e^{-u}, & \text{if } 0 < u < 1. \end{cases} \end{aligned}$$

One remarks: one can also use transformation and define for example $U = X+Y$ and $V = Y$, then find the joint density $f_{U,V}(u,v)$ of (U,V), and derive the marginal density $f_U(u)$. □

2 (3 points)

Let $(X,Y)'$ have a joint probability density function as follows

$$f(x,y) = \begin{cases} c \cdot x \cdot y, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2.1) (1p) Find the value of c such that $f(x,y)$ is indeed a density function.
(2.2) (1p) Compute the conditional expectation $E(Y|X=x)$ for $0 < x < 1$.
(2.3) (1p) Compute the conditional expectation $E(X|Y=y)$ for $0 < y < 1$.

Solution. (2.1)

$$1 = \int_0^1 \left(\int_0^x c \cdot x \cdot ydy \right) dx = \int_0^1 c \cdot x \cdot \left(\int_0^x ydy \right) dx = \int_0^1 c \cdot x \cdot x^2/2dx = c/8 \quad \Longrightarrow \quad c = 8.$$

(2.2) The marginal probability density function is

$$f_X(x) = \int_0^x c \cdot x \cdot ydy = cx^3/2 \text{ for } 0 < x < 1.$$

Therefore, the conditional probability density function is

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{c \cdot x \cdot y}{cx^3/2} = \frac{2y}{x^2}, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The conditional expectation can be then computed as

$$E(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy = \int_0^x y\frac{2y}{x^2}dy = \frac{2}{x^2}\int_0^x y^2dy = \frac{2x}{3}.$$

(2.3) The marginal probability density function is

$$f_Y(y) = \int_y^1 c \cdot x \cdot y dx = cy(1-y^2)/2 \text{ for } 0 < y < 1.$$

Therefore, the conditional probability density function is

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{c \cdot x \cdot y}{cy(1-y^2)/2} = \frac{2x}{(1-y^2)}, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The conditional expectation can be then computed as

$$E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y=y}(x) dx = \int_y^1 x \frac{2x}{(1-y^2)} dx = \frac{2}{(1-y^2)} \int_y^1 x^2 dx = \frac{2}{3} \cdot \frac{1-y^3}{1-y^2}.$$

□

3 (3 points)

Let the probability generating function $g_{X,Y}(s,t)$ of $(X,Y)'$ be given as

$$g_{X,Y}(s,t) = E(s^X t^Y) = \exp\{(s-1) + 2(t-1) + 3(st-1)\}.$$

(3.1) (1p) Find the probability generating function $g_X(s)$ of X and $P(X=n)$ for $n \geq 0$.
(3.2) (1p) Find the probability generating function $g_Y(t)$ of Y and $P(Y=n)$ for $n \geq 0$.
(3.3) (1p) Find the probability generating function $g_{X+Y}(u)$ of $X+Y$.

Solution. (3.1) The probability generating function $g_X(s)$ of X is

$$g_X(s) = E(s^X) = g_{X,Y}(s,1) = \exp\{(s-1) + 3(s-1)\} = \exp\{4(s-1)\}.$$

Therefore,

$$P(X=n) = \frac{g_X^{(n)}(0)}{n!} = \frac{4^n e^{-4}}{n!}.$$

(3.2) The probability generating function $g_Y(t)$ of Y is

$$g_Y(t) = E(t^Y) = g_{X,Y}(1,t) = \exp\{2(t-1) + 3(t-1)\} = \exp\{5(t-1)\}.$$

Therefore,

$$P(Y=n) = \frac{g_Y^{(n)}(0)}{n!} = \frac{5^n e^{-5}}{n!}.$$

(3.3) The probability generating function $g_{X+Y}(u)$ of $X+Y$ is

$$g_{X+Y}(u) = E(u^{X+Y}) = g_{X,Y}(u,u) = \exp\{(u-1) + 2(u-1) + 3(u^2-1)\} = \exp\{3(u-1) + 3(u^2-1)\}.$$

□

4 (3 points)

Suppose that X_1, X_2, X_3 and X_4 are independent $U(0,1)$ random variables, and let $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})$ be the corresponding order statistic. Find the probability $P(X_{(3)} + X_{(4)} \leq 1)$.

Solution. It is from Theorem 3.1 (p.110 book) that the joint probability density function of $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})$ is

$$f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(x_1, x_2, x_3, x_4) = 4! = 24, \qquad 0 < x_1 < x_2 < x_3 < x_4 < 1.$$

Therefore, the joint probability density function of $(X_{(3)}, X_{(4)})$ is

$$\begin{aligned} f_{X_{(3)},X_{(4)}}(x_3,x_4) &= \int_0^{x_3}\left(\int_{x_1}^{x_3} f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(x_1,x_2,x_3,x_4)dx_2\right)dx_1 = \int_0^{x_3}\left(\int_{x_1}^{x_3} 24dx_2\right)dx_1 \\ &= \int_0^{x_3}(24x_3 - 24x_1)\,dx_1 = 24x_3^2 - 12x_3^2 = 12x_3^2, \text{ for } 0 < x_3 < x_4 < 1. \end{aligned}$$

Therefore, by drawing the region of (x_3, x_4),

$$\begin{aligned} P(X_{(3)} + X_{(4)} \le 1) &= \int_0^{1/2}\left(\int_{x_3}^{1-x_3} f_{X_{(3)},X_{(4)}}(x_3,x_4)dx_4\right)dx_3 = \int_0^{1/2}\left(\int_{x_3}^{1-x_3} 12x_3^2 dx_4\right)dx_3 \\ &= \int_0^{1/2}\left(12x_3^2 - 24x_3^3\right)dx_3 = \frac{1}{8}. \end{aligned}$$

□

5 (3 points)

Let X and Y be two random variables such that $X \sim N(3, 4^2)$ and $Y|X = x \sim N(10 + 20x, 5^2)$ (that is, the conditional distribution of Y given $X = x$ is $N(10 + 20x, 5^2)$). Find the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\mathbf{C}$ of the two dimensional random variable $(X, Y)'$.

Solution. The mean of X is directly from the problem $E(X) = 3$. The mean of Y can be computed as

$$E(Y) = E(E(Y|X)) = E(10 + 20X) = 10 + 20E(X) = 10 + 20 \cdot 3 = 70.$$

Therefore the mean vector is $\boldsymbol{\mu} = (3, 70)'$
For the covariance matrix, it is known that $V(X) = 4^2 = 16$. The variance of Y can be computed as

$$V(Y) = E(V(Y|X)) + V(E(Y|X)) = E(5^2) + V(10 + 20X) = 25 + 400V(X) = 25 + 400 \cdot 4^2 = 6425.$$

The covariance is computed as

$$\begin{aligned} cov(X,Y) &= E(XY) - E(X)E(Y) = E(E(XY|X)) - 3 \cdot 70 = E(XE(Y|X)) - 210 = E(X(10 + 20X))) - 210 \\ &= E(10X + 20X^2) - 210 = 10E(X) + 20E(X^2) - 210 \\ &= 10 \cdot 3 + 20(3^2 + 4^2) - 210 = 30 + 20 \cdot 25 - 210 = 320. \end{aligned}$$

Therefore the covariance matrix is

$$\mathbf{C} = \begin{pmatrix} 16 & 320 \\ 320 & 6425 \end{pmatrix}.$$

□

6 (3 points)

Let $X_n \sim Bin(n^2, 1/n)$. Use convergence of moment generating functions to show that

$$\frac{X_n - n}{\sqrt{n}} \xrightarrow{d} N(0,1), \quad \text{as } n \to \infty.$$

(Hint: moment generating function of Binomial random variable is $\psi_{Bin(n,p)}(t) = [(1-p) + pe^t]^n$, and moment generating function of standard normal random variable is $\psi_{N(0,1)}(t) = e^{t^2/2}$. You might also need to use the expansions $e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$)

Solution. The moment generating function of $\frac{X_n - n}{\sqrt{n}}$ is

$$\begin{aligned}
\psi_{\frac{X_n-n}{\sqrt{n}}}(t) &= E\exp\{t\cdot\frac{X_n-n}{\sqrt{n}}\} = e^{-t\sqrt{n}}\cdot E\exp\{\frac{t}{\sqrt{n}}X_n\} = e^{-t\sqrt{n}}\cdot\psi_{X_n}(\frac{t}{\sqrt{n}}) \\
&= e^{-t\sqrt{n}}\cdot\left[(1-\frac{1}{n})+\frac{1}{n}e^{\frac{t}{\sqrt{n}}}\right]^{n^2} = e^{-t\sqrt{n}}\cdot\left[1+\frac{1}{n}(e^{\frac{t}{\sqrt{n}}}-1)\right]^{n^2} \\
&= e^{-t\sqrt{n}}\cdot\exp\left\{n^2\ln\left(1+\frac{1}{n}(e^{\frac{t}{\sqrt{n}}}-1)\right)\right\} \quad (\text{use } e^x-1=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots) \\
&= e^{-t\sqrt{n}}\cdot\exp\left\{n^2\ln\left(1+\frac{1}{n}(\frac{t}{\sqrt{n}}+\frac{t^2}{2n}+o(\frac{1}{n}))\right)\right\} \\
&= e^{-t\sqrt{n}}\cdot\exp\left\{n^2\ln\left(1+(\frac{t}{n^{3/2}}+\frac{t^2}{2n^2}+o(\frac{1}{n^2}))\right)\right\} \quad (\text{ use } \ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\dots) \\
&= e^{-t\sqrt{n}}\cdot\exp\left\{n^2\left(\frac{t}{n^{3/2}}+\frac{t^2}{2n^2}+o(\frac{1}{n^2})\right)\right\} \\
&= e^{-t\sqrt{n}}\cdot\exp\left\{t\sqrt{n}+\frac{t^2}{2}+o(1)\right\} = \exp\left\{\frac{t^2}{2}+o(1)\right\} \to e^{t^2/2} = \psi_{N(0,1)}(t),
\end{aligned}$$

which completes the proof. □

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a)=1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1)=p(1)=\frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0\le p\le 1$	$p(0)=q,\ p(1)=p;\ q=1-p$	p	pq	$q+pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n=1,2,\ldots$, $0\le p\le 1$	$p(k)=\binom{n}{k}p^kq^{n-k},\ k=0,1,\ldots,n;\ q=1-p$	np	npq	$(q+pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0\le p\le 1$	$p(k)=pq^k,\ k=0,1,2,\ldots;\ q=1-p$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$	$\dfrac{p}{1-qe^{it}}$
First success $\mathrm{Fs}(p)$, $0\le p\le 1$	$p(k)=pq^{k-1},\ k=1,2,\ldots;\ q=1-p$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	$\dfrac{pe^{it}}{1-qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n=1,2,3,\ldots$, $0\le p\le 1$	$p(k)=\binom{n+k-1}{k}p^nq^k,\ k=0,1,2,\ldots;$ $q=1-p$	$n\dfrac{q}{p}$	$n\dfrac{q}{p^2}$	$(\frac{p}{1-qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m>0$	$p(k)=e^{-m}\dfrac{m^k}{k!},\ k=0,1,2,\ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N,n,p)$, $n=0,1,\ldots,N$, $N=1,2,\ldots$, $p=0,\frac{1}{N},\frac{2}{N},\ldots,1$	$p(k)=\dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k=0,1,\ldots,Np;$ $q=1-p;$ $n-k=0,\ldots,Nq$	np	$npq\,\dfrac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$, $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a>0$	$f(x)=\frac{1}{a}e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\frac{1}{\Gamma(p)}x^{p-1}\frac{1}{a^p}e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\ldots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}x^{r-1}(1-x)^{s-1}$, $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)$ $-\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2},\ n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2},$ $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2,$ $n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x)=\frac{1}{\pi}\cdot\frac{a}{a^2+(x-m)^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{imt-a\|t\|}$
$C(0,1)$	$f(x)=\frac{1}{\pi}\cdot\frac{1}{1+x^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{-\|t\|}$
Pareto	$f(x)=\frac{\alpha k^\alpha}{x^{\alpha+1}},\ x>k$	$\frac{\alpha k}{\alpha-1},\ \alpha>1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha>2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k>0,\ \alpha>0$				