Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '

1 (3 points)

Let a two dimensional random vector $(X, Y)'$ have a joint probability density function as follows

$$f(x,y) = \begin{cases} e^{-x^2 y}, & \text{if } x \geq 1 \text{ and } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability density function of X^2Y.

Solution. Let $U = X^2Y$ and $V = Y$. Then it is clear that $U \geq V > 0$, and

$$X = \sqrt{U/V}, \quad Y = V, \qquad J = |\frac{\partial(x \;\; y)}{\partial(u \;\; v)}| = \frac{1}{2}u^{-1/2}v^{-1/2}.$$

Therefore the joint probability density function of $(U, V)'$ is

$$f_{U,V}(u,v) = f(x^{-1}(u,v), y^{-1}(u,v))|J| = f(\sqrt{u/v}, v)|J| \begin{cases} e^{-u} \cdot \frac{1}{2}u^{-1/2}v^{-1/2}, & \text{if } u \geq v > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The marginal probability density function of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)dv = \int_0^u e^{-u} \cdot \frac{1}{2}u^{-1/2}v^{-1/2}dv = e^{-u} \cdot \frac{1}{2}u^{-1/2} \int_0^u v^{-1/2}dv = e^{-u}, u > 0.$$

□

2 (3 points)

Let a two dimensional random vector $(X, Y)'$ have a joint probability density function as follows

$$f(x,y) = \begin{cases} 2, & \text{if } x \geq 0, y \geq 0 \text{ and } x + y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2.1) (1p) Find the marginal probability density function $f_X(x)$ of X.
(2.2) (2p) Compute the conditional expectation $E(Y|X = x)$.

Solution. (2.1) The marginal probability density function is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{1-x} 2dx = 2(1-x), \quad 0 \leq x \leq 1.$$

(2.2) In order to compute $E(Y|X = x)$, the conditional probability density function is

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{1}{1-x}, \quad x \geq 0, y \geq 0 \text{ and } x + y \leq 1.$$

The conditional expectation can be then computed as

$$E(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy = \int_0^{1-x} \frac{y}{1-x}dy = \frac{1}{1-x}\frac{1}{2}(1-x)^2 = \frac{1-x}{2}, \quad 0 \leq x \leq 1.$$

□

3 (3 points)

Let $X_1 \sim Exp(1), X_2 \sim Exp(1), \ldots, X_n \sim Exp(1), \ldots$ be a sequence of independent exponential random variables. Let

$$S_N = X_1 + X_2 + \ldots + X_N,$$

where $N \sim Po(10)$ is a Poisson random variable which is independent of $X_1, X_2, \ldots$. When $N = 0$, we define $S_0 = 0$.
(3.1) (1p) Find the moment generating function $\psi_{S_N}(t)$ of S_N. (Hint: Moment generating function $\psi_X(t) = E(e^{tX})$)
(3.2) (1p) Find the first moment $E(S_N)$ of S_N.
(3.3) (1p) Find the second moment $E(S_N^2)$ of S_N.

Solution. (3.1) It is from Theorem 6.3 (Page. 84 book) that the moment generating function is

$$\psi_{S_N}(t) = g_N(\psi_X(t)) = e^{10(\frac{1}{1-t}-1)}, \text{ for } t < 1.$$

where the probability generating function $g_N(t)$ of N is (see Page. 63 book)

$$g_N(t) = e^{10(t-1)},$$

and the moment generating function $\psi_X(t)$ of each X_i is (see Page. 67 book)

$$\psi_X(t) = \frac{1}{1-t}, \text{ for } t < 1.$$

(3.2) It is from Theorem 3.3 (Page. 64 book) that

$$E(S_N) = \psi'_{S_N}(0) = \left[e^{-10}e^{10(1-t)^{-1}} \cdot 10(1-t)^{-2}\right]_{t=0} = 10.$$

(3.3) It is from Theorem 3.3 (Page. 64 book) that

$$E(S_N^2) = \psi''_{S_N}(0) = \left[10e^{-10}(e^{10(1-t)^{-1}} \cdot 10(1-t)^{-4} + e^{10(1-t)^{-1}} \cdot 2(1-t)^{-3})\right]_{t=0} = 120.$$

□

4 (3 points)

Suppose that the running times (in seconds) in a 100-meter LiU race are distributed as $U(10.0, 16.0)$ (namely, an uniform random variable on the interval $(10.0, 16.0)$). Suppose that there are 6 competitors in a 100-meter LiU race, find the probability that the winner is at most 3 seconds faster than the slowest runner?

Solution. Let $X_1, X_2, \ldots, X_6$ denote the running times of these 6 competitors, then the winner is $X_{(1)} = \min\{X_1, X_2, \ldots, X_6\}$, and the slowest runner is $X_{(6)} = \max\{X_1, X_2, \ldots, X_6\}$. Therefore the probability that the winner is at most 3 seconds faster than the slowest runner $= P(X_{(6)} - X_{(1)} \leq 3)$.
Recall the definition "Range" $R_6 := X_{(6)} - X_{(1)}$, it is from Theorem 2.2 (Page. 106 book) that the probability density function of R_6 is (it is clear that $f_{R_6}(r) = 0$ when $r \geq 6$),

$$f_{R_6}(r) = 6(6-1)\int_{-\infty}^{\infty}(F(u+r) - F(u))^4 f(u+r)f(u)du, \qquad 6 > r > 0.$$

Note that $f(x) = \frac{1}{6}$ for $10 < x < 16$ and

$$F(x) = \begin{cases} 0, & \text{if } x \leq 10, \\ \frac{x-10}{6}, & \text{if } 10 < x < 16, \\ 1, & \text{if } x \geq 16. \end{cases}$$

Therefore, for $0 < r < 6$,

$$f_{R_6}(r) = 6(6-1)\int_{-\infty}^{\infty}(F(u+r) - F(u))^4 f(u+r)f(u)du = 6 \cdot 5\int_{10}^{16}(F(u+r) - \frac{u-10}{6})^4 f(u+r)\frac{1}{6}du$$

$$(\text{with } v = u+r) = 6 \cdot 5\int_{10+r}^{16+r}(F(v) - \frac{(v-r)-10}{6})^4 f(v)\frac{1}{6}dv = 6 \cdot 5\int_{10+r}^{16}(F(v) - \frac{(v-r)-10}{6})^4 \cdot \frac{1}{6}\frac{1}{6}dv$$

$$= 6 \cdot 5\int_{10+r}^{16}(\frac{v-10}{6} - \frac{(v-r)-10}{6})^4 \cdot \frac{1}{6}\frac{1}{6}dv = 6 \cdot 5 \cdot \frac{1}{6^2}\int_{10+r}^{16}(\frac{r}{6})^4 dv = \frac{5r^4}{6^5}(6-r).$$

So,

$$
\begin{aligned}
&P(\text{the winner is at most 3 seconds faster than the slowest runner}) = P(R_6 \leq 3)\\
&= \int_0^3 f_{R_6}(r)dr = \int_0^3 \frac{5r^4}{6^5}(6-r)dr = \frac{1}{6^4}\int_0^3 5r^4 dr - \frac{5}{6^6}\int_0^3 6r^5 dr\\
&= \frac{1}{6^4}\cdot 3^5 - \frac{5}{6^6}\cdot 3^6 = \frac{7}{64} = 0.109375.
\end{aligned}
$$

□

5 (3 points)

Let $(X,Y)'$ be two dimensional normal random vector. Suppose that the variance $V(X)$ of X is equal to the variance $V(Y)$ of Y. Are $X-Y$ and $X+Y$ independent random variables? Why?

Solution. Step 1: Since $(X,Y)'$ is a two dimensional normal random vector, it is from "Definition I" and "Theorem 3.1" (Page. 121 book) that $(X-Y, X+Y)'$ is also a two dimensional normal random vector.

Step 2: Since $(X-Y, X+Y)'$ is a two dimensional normal random vector, the independence of $X-Y$ and $X+Y$ is equivalent to $cov(X-Y, X+Y) = 0$.

Step 3: The covariance can be computed as

$$cov(X-Y,X+Y) = cov(X,X)+cov(X,Y)-cov(Y,X)-cov(Y,Y) = cov(X,X)-cov(Y,Y) = V(X)-V(Y) = 0.$$

Theretofore, Yes, $X-Y$ and $X+Y$ are independent!

□

6 (3 points)

(6.1) (1p) Let $\{X_1, X_2, \ldots, X_n, \ldots\}$ be a sequence of random variables with

$$P(X_n = \pi) = 1 - \frac{1}{\sqrt{n}}, \qquad P(X_n = n) = \frac{1}{\sqrt{n}}, \qquad \text{for } n \geq 1.$$

Prove that X_n converge to π in probability.
(6.2) (2p) Let $\{Y_1, Y_2, \ldots, Y_n, \ldots\}$ be a sequence of random variables with

$$P(Y_n = \pi) = 1 - \frac{1}{n^2}, \qquad P(Y_n = n) = \frac{1}{n^2}, \qquad \text{for } n \geq 1.$$

Prove that Y_n converge to π almost surely.

Solution. (6.1) For any $\epsilon > 0$, it follows that for large n,

$$P(|X_n - \pi| < \epsilon) = P(X_n = \pi) = 1 - \frac{1}{\sqrt{n}} \to 0, \text{ as } n \to \infty,$$

which proves that X_n converge to π in probability.
(6.2) For any $\epsilon > 0$, let us consider the events $\{|Y_n - \pi| > \epsilon\}_{n\geq 1}$. The fact that

$$\sum_{n=1}^{\infty} P(|Y_n - \pi| > \epsilon) = (\text{or} \leq) \sum_{n=1}^{\infty} P(Y_n = n) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

implies (based on Borel-Cantelli lemma, Theorem 7.1, Page. 205 book) that $P(|Y_n - \pi| > \epsilon \text{ i.o.}) = 0$, which is equivalent to $Y_n \to \pi$ almost surely (see statement (7.2) Page. 205 book). □

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	Var X	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\text{Be}(p)$, $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\text{Bin}(n, p)$, $n = 1, 2, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\text{Ge}(p)$, $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$	$\dfrac{p}{1 - qe^{it}}$
First success $\text{Fs}(p)$, $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	$\dfrac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\text{NBin}(n, p)$, $n = 1, 2, 3, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\dfrac{q}{p}$	$n\dfrac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson $\text{Po}(m)$, $m > 0$	$p(k) = e^{-m}\dfrac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it} - 1)}$
Hypergeometric $H(N, n, p)$, $n = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $p = 0, \dfrac{1}{N}, \dfrac{2}{N}, \ldots, 1$	$p(k) = \dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\dfrac{N - n}{N - 1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$, $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a>0$	$f(x)=\frac{1}{a}\,e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\frac{1}{\Gamma(p)}\,x^{p-1}\,\frac{1}{a^p}\,e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\ldots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1}$, $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^\beta\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)$ $-\Gamma(\beta+1)^2\big)$	$*$
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	$*$
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	$*$
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2}$, $n>2$	$*$
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2}$, $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\left(\frac{n}{n-2}\right)^2,$ $n>4$	$*$

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\|t\|}$
$C(0,1)$	$f(x) = \dfrac{1}{\pi} \cdot \dfrac{1}{1+x^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\|t\|}$
Pareto	$f(x) = \dfrac{\alpha k^\alpha}{x^{\alpha+1}},\ x > k$	$\dfrac{\alpha k}{\alpha-1},\ \alpha > 1$	$\dfrac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$				