Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '

1 (3 points)

Let $X \sim U(0,1)$ and $Y \sim U(0,2)$ be two independent uniform random variables. Find the density function of $U = X + Y$. (Hint: You can use either convolution formula or transformation theorem. Be really really careful with the bounds of each variable!!! It might help to draw a graph for the bounds)

2 (3 points)

Let us throw a fair die twice independently. Set $U =$ the outcome of the first throw and $V =$ the outcome of the second throw. Define

$$X = U \qquad \text{and} \qquad Y = U + V.$$

Find the conditional expectation $E(Y|X = x)$.

3 (3 points)

Consider the following situation: Hanna has a coin with $P(\text{head}) = p_1$ and Livia has a coin with $P(\text{head}) = p_2$. Hanna tosses her coin m times. Each time Hanna obtains "head", Livia tosses her coin (otherwise not). Let X be the total number of heads obtained by Livia. Then X can be modeled as follows:

$$X|N = n \sim Bin(n, p_2), \quad \text{with } N \sim Bin(m, p_1), \quad 0 < p_1, p_2 < 1,$$

where N denotes the total number of heads obtained by Hanna. Find the probability generating function (PGF) of X. Do you recognize the distribution of X?
(Hint: probability generating function of a Binomial random variable is $g_{Bin(n,p)}(t) = (q + pt)^n$ with $q = 1 - p$)

4 (3 points)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $Exp(1)$ random variables, and $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be the order statistic. Define

$$Y_1 = X_{(1)}, \quad Y_k = X_{(k)} - X_{(k-1)}, \text{ for } k = 2, 3, \ldots, n.$$

(4.1) (1p) Find the joint density function $f_{X_{(1)},X_{(2)},\ldots,X_{(n)}}(x_1, x_2, \ldots, x_n)$ of $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$.
(4.2) (1p) Find the joint density function $f_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n)$ of $(Y_1, Y_2, \ldots, Y_n)$.
(4.3) (1p) find the density function $f_{Y_n}(y_n)$ of Y_n.

5 (3 points)

Let $(X_1, X_2)'$ be two dimensional random vector whose characteristic function is given as follows:

$$\varphi_{X_1,X_2}(t_1, t_2) = e^{it_1 - 2t_1^2 - t_2^2 - t_1 t_2},$$

where i is the imaginary unit.
(5.1) (2p) Is $(X_1, X_2)'$ a two dimensional normal random vector? If yes, specify the mean vector μ and the covariance matrix Λ. If no, specify the reason(s).
(5.2) (1p) Find the distribution of $X_1 + X_2$. (Namely, specify which distribution with which parameters)

6 (3 points)

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with a common distribution function $F(x)$ (which is $F(x) = P(X_i \leq x)$). Let $F_n(x)$ be the empirical distribution function defined as

$$F_n(x) = \frac{\#\text{ observations among } X_1, X_2, \ldots, X_n \leq x}{n}.$$

For example, if we have observed $\{2, 3, 5, 4\}$ for $\{X_1, X_2, X_3, X_4\}$ then $F_4(2.5) = \frac{1}{4}$ and $F_4(3.2) = \frac{2}{4}$.
(6.1) (1p) For each fixed x, prove that $F_n(x)$ converge to $F(x)$ in probability as $n \to \infty$.
(6.2) (2p) For each fixed x, determine $a(x)$ and $b(x)$, and show the following convergence in distribution

$$\frac{F_n(x) - a(x)}{b(x)/\sqrt{n}} \xrightarrow{d} N(0,1), \qquad \text{as } n \to \infty.$$

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\mathrm{Bin}(n, p)$, $n = 1, 2, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$	$\dfrac{p}{1 - qe^{it}}$
First success $\mathrm{Fs}(p)$, $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	$\dfrac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\mathrm{NBin}(n, p)$, $n = 1, 2, 3, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\dfrac{q}{p}$	$n\dfrac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m > 0$	$p(k) = e^{-m}\dfrac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it} - 1)}$
Hypergeometric $H(N, n, p)$, $n = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $p = 0, \dfrac{1}{N}, \dfrac{2}{N}, \ldots, 1$	$p(k) = \dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\ \ k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\,\dfrac{N - n}{N - 1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$ $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a>0$	$f(x)=\frac{1}{a}e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\frac{1}{\Gamma(p)}x^{p-1}\frac{1}{a^p}e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\dots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}x^{r-1}(1-x)^{s-1},$ $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1) -\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},\ -\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},\ -\infty<x<\infty$	0	$\dfrac{n}{n-2},\ n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},\ x>0$	$\frac{n}{n-2},\ n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2,\ n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\|t\|}$
$C(0,1)$	$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\|t\|}$
Pareto	$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}},\ x > k$	$\frac{\alpha k}{\alpha - 1},\ \alpha > 1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$				