TAMS46: Probability Theory (Second Course) | Provkod: TEN1 | 09 January 2025, 14:00-18:00

Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '

1 (3 points)

Let $X \sim Exp(1)$ and $Y \sim Exp(1)$ be two independent exponential random variables.
(1.1) (2p) Find the conditional density function $f_{X|X+Y=2}(x)$ of X given that $X+Y=2$.
(1.2) (1p) Find the conditional expectation $E(X|X+Y=2)$.

Solution. (1.1) Let $U = X$ and $V = X + Y$. Then

$$X = U, \quad Y = V - U, \qquad J = |\frac{\partial(x \;\; y)}{\partial(u \;\; v)}| = 1, \quad 0 < u < v.$$

Therefore the joint probability density function of $(U, V)'$ is, based on $f_{X,Y}(x, y) = e^{-x} \cdot e^{-y}$,

$$f_{U,V}(u, v) = f(x^{-1}(u, v), y^{-1}(u, v))|J| = e^{-u} \cdot e^{-(v-u)} = e^{-v}, 0 < u < v.$$

The density function $f_V(v)$ of V is then

$$f_V(v) = \int_0^v f_{U,V}(u, v)du = \int_0^v e^{-v}du = v \cdot e^{-v}, v > 0.$$

Therefore, the conditional density function $f_{X|X+Y=2}(x)$ is

$$f_{X|X+Y=2}(x) = \frac{f_{U,V}(x, 2)}{f_V(2)} = \frac{e^{-2}}{2 \cdot e^{-2}} = \frac{1}{2}, \qquad 0 < x < 2.$$

(1.2)

$$E(X|X+Y=2) = \int_0^2 x \cdot f_{X|X+Y=2}(x)dx = \int_0^2 x \cdot \frac{1}{2}dx = 1.$$

□

2 (3 points)

Let $X \sim U(0, 1)$ be an uniform random variable. Let Y be a random variable depending on X in the following way:

$$Y|X = x \sim U(0, 1 - x), \quad 0 < x < 1.$$

(2.1) (1p) Find $E(Y)$.
(2.2) (2p) Find $E(X \cdot Y)$.

Solution. (2.1) It is from $Y|X = x \sim U(0, 1-x)$ that $E(Y|X) = \frac{1-X}{2}$, which implies that

$$E(Y) = E(E(Y|X)) = E(\frac{1-X}{2}) = \frac{1}{2}E(1-X) = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}.$$

(2.2)

$$E(X \cdot Y) = E(E(X \cdot Y|X)) = E(X \cdot E(Y|X)) = E(X \cdot \frac{1-X}{2}) = \frac{1}{2}(E(X) - E(X^2)) = \frac{1}{2}(\frac{1}{2} - \frac{1}{3}) = \frac{1}{12}.$$

□

3 (3 points)

A Galton-Watson process starts with one individual who reproduces according to the following principle:

# of children Y	0	1	2
probability $p(y)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

The children reproduce according to the same rule, independently of each other, and so on. Find the probability of extinction.

Solution. According to Section 3.7 on the book, the probability of extinction is the smallest nonnegative root of the equation

$$t = g(t)$$

where $g(t)$ is the probability generating function of Y, that is

$$g(t) = E(t^Y) = t^0 \cdot P(Y=0) + t^1 \cdot P(Y=1) + t^2 \cdot P(Y=2) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2.$$

The equation $t = g(t)$ has two roots: $t = 1$ and $t = 1/2$. Therefore the probability of extinction is $1/2$.

□

4 (3 points)

Let $X_1, X_2, \dots$ be i.i.d. $U(0,1)$ uniform random variables. Show that
(4.1) (1.5p) $\max_{1\le k\le n} X_k \xrightarrow{p} 1$, as $n \to \infty$.
(4.2) (1.5p) $\min_{1\le k\le n} X_k \xrightarrow{p} 0$, as $n \to \infty$.

Solution. (4.1) For any small $\epsilon > 0$, it holds that

$$\begin{aligned} P(|\max_{1\le k\le n} X_k - 1| > \epsilon) &= P(\max_{1\le k\le n} X_k > 1+\epsilon) + P(\max_{1\le k\le n} X_k < 1-\epsilon) = 0 + P(\max_{1\le k\le n} X_k < 1-\epsilon) \\ &= P(X_1 < 1-\epsilon, X_2 < 1-\epsilon, \dots, X_n < 1-\epsilon) = (P(X_1 < 1-\epsilon))^n = (1-\epsilon)^n \to 0, \end{aligned}$$

which proves that $\max_{1\le k\le n} X_k \xrightarrow{p} 1$, as $n \to \infty$.
(4.2) For any small $\epsilon > 0$, it holds that

$$\begin{aligned} P(|\min_{1\le k\le n} X_k| > \epsilon) &= P(\min_{1\le k\le n} X_k > \epsilon) + P(\min_{1\le k\le n} X_k < -\epsilon) = P(\min_{1\le k\le n} X_k > \epsilon) + 0 \\ &= P(X_1 > \epsilon, X_2 > \epsilon, \dots, X_n > \epsilon) = (P(X_1 > \epsilon))^n = (1-\epsilon)^n \to 0, \end{aligned}$$

which proves that $\min_{1\le k\le n} X_k \xrightarrow{p} 0$, as $n \to \infty$.

□

5 (3 points)

Let X_1, X_2 and X_3 be i.i.d. $N(2,1)$ normal random variables. Find the distribution of $X_1 + 3X_2 - 2X_3$ given that $2X_1 - X_2 = 1$.

Solution. Let us define $Y_1 = X_1 + 3X_2 - 2X_3$ and $Y_2 = 2X_1 - X_2$. As (X_1, X_2, X_3) is a 3-dim normal random vector, we know that (Y_1, Y_2) is a 2-dim normal random vector. To obtain its mean vector μ_Y and covariance matrix Λ_Y, we compute the following:

$$\begin{aligned} E(Y_1) &= E(X_1 + 3X_2 - 2X_3) = E(X_1) + 3E(X_2) - 2E(X_3) = 4, \\ E(Y_2) &= E(2X_1 - X_2) = 2E(X_1) - E(X_2) = 2, \\ V(Y_1) &= V(X_1 + 3X_2 - 2X_3) = V(X_1) + 3^2 V(X_2) + (-2)^2 V(X_3) = 14, \\ V(Y_2) &= V(2X_1 - X_2) = 2^2 V(X_1) + (-1)^2 V(X_2) = 5, \\ cov(Y_1, Y_2) &= E[(Y_1 - E(Y_1))(Y_2 - E(Y_2))] = E[(X_1 + 3X_2 - 2X_3 - 4)(2X_1 - X_2 - 2)] = -1. \end{aligned}$$

Therefore,

$$\mu_Y = (4,2)', \qquad \Lambda_Y = \begin{pmatrix} 14 & -1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} (\sqrt{14})^2 & \frac{-1}{\sqrt{14}\sqrt{5}} \cdot \sqrt{14}\sqrt{5} \\ \frac{-1}{\sqrt{14}\sqrt{5}} \cdot \sqrt{14}\sqrt{5} & (\sqrt{5})^2 \end{pmatrix}$$

where the correlation coefficient is $\rho = \frac{-1}{\sqrt{14}\sqrt{5}}$. Then according to Book: Section 5.6 #(6.2),

$$Y_1|Y_2 = 1 \sim N(\mu, \sigma^2),$$

where $\mu = 4 + \rho\frac{\sigma_{Y_1}}{\sigma_{Y_2}}(1-2) = 4 + \frac{1}{5} = 4.2$, and $\sigma^2 = 14(1-\rho^2) = \frac{69}{5} = 13.8$.

□

6 (3 points)

Let $X_1, X_2, \ldots$ be i.i.d. $L(a)$ Laplace random variables, and let $N \sim Po(m)$ be independent of $X_1, X_2, \ldots$. Define $S_N = X_1 + X_2 + \ldots + X_N$ (where $S_0 = 0$).
(6.1) (2p) Find the characteristic function $\varphi_{S_N}(t)$ of S_N.
(6.2) (1p) Find the limit distribution of S_N as $m \to \infty$ and $a \to 0$ in such a way that $m \cdot a^2 \to 1$. (Hint: limit of $\varphi_{S_N}(t)$)

Solution. (6.1) It is noted that $\varphi_X(t) = \frac{1}{1+a^2t^2}$, therefore

$$\varphi_{S_N}(t) = E(e^{itS_N}) = g_N(\varphi_X(t)) = e^{m(\varphi_X(t)-1)} = e^{m(\frac{1}{1+a^2t^2}-1)} = e^{-\frac{ma^2t^2}{1+a^2t^2}}.$$

(6.2) Taking into account $m \to \infty$ and $a \to 0$ in such a way that $m \cdot a^2 \to 1$, it follows that

$$\varphi_{S_N}(t) = e^{-\frac{ma^2t^2}{1+a^2t^2}} \to e^{-t^2} = \varphi_{N(0,2)}(t).$$

That is $S_N \xrightarrow{d} N(0,2)$.

□

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	Var X	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli Be(p), $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial Bin(n,p), $n = 1, 2, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric Ge(p), $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1 - qe^{it}}$
First success Fs(p), $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1 - qe^{it}}$
Negative binomial NBin(n,p), $n = 1, 2, 3, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson Po(m), $m > 0$	$p(k) = e^{-m}\frac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it} - 1)}$
Hypergeometric $H(N, n, p)$, $n = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $p = 0, \frac{1}{N}, \frac{2}{N}, \ldots, 1$	$p(k) = \dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\ k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\,\frac{N - n}{N - 1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\operatorname{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$, $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\operatorname{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\operatorname{Exp}(a),\ a>0$	$f(x)=\frac{1}{a}\,e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\frac{1}{\Gamma(p)}\,x^{p-1}\,\frac{1}{a^p}\,e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\ldots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1}$, $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)$ $-\Gamma(\beta+1)^2\big)$	$*$
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	$*$
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	$*$
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2},\ n>2$	$*$
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2},$ $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2,$ $n>4$	$*$

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+(x-m)^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{imt-a\|t\|}$
$C(0,1)$	$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2},\ -\infty < x < \infty$	$\nexists$	$\nexists$	$e^{-\|t\|}$
Pareto	$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}},\ x > k$	$\frac{\alpha k}{\alpha - 1},\ \alpha > 1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha > 2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k > 0,\ \alpha > 0$				