TAMS46: Probability Theory (Second Course) | Provkod: TEN1 | 25 August 2025, 08:00-12:00

Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '

1 (3 points)

Let X be a continuous random variable with a probability density function $f_X(x) = \frac{1}{2}e^{-|x|}$ for $-\infty < x < \infty$. Define $Y = X^2$. Find the density function $f_Y(y)$ of Y.

Solution. **Method-1:** It is from many-to-one transformation theorem that

$$f_Y(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}} = \frac{1}{2}e^{-\sqrt{y}}\frac{1}{\sqrt{y}}, \qquad y \geq 0.$$

Method-2: Direct computations yield: for $y \geq 0$,

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) + P(X \geq -\sqrt{y}) = F_X(\sqrt{y}) + (1 - F_X(-\sqrt{y}))$$

Taking derivative gives

$$f_Y(y) = F'_Y(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}} = \frac{1}{2}e^{-\sqrt{y}}\frac{1}{\sqrt{y}}, \qquad y \geq 0.$$

□

2 (3 points)

Let $Y = Z \cdot X$ where $Z \sim N(\mu, \sigma^2)$ (normal) and $X \sim Be(p)$ (Bernoulli) are two independent random variables.
(2.1) (1p) Find the expectation $E(Y)$.
(2.2) (1p) Find the expectation $E(X \cdot Y \cdot Z)$.
(2.3) (1p) Find the conditional expectation $E(Y|X)$.

Solution. (2.1)

$$E(Y) = E(Z \cdot X) = (\text{independence of } Z \text{ and } X) = E(Z) \cdot E(X) = \mu \cdot p.$$

(2.2)

$$E(X \cdot Y \cdot Z) = E(X \cdot Z \cdot X \cdot Z) = E(X^2 \cdot Z^2) = E(X^2) \cdot E(Z^2) = p \cdot (\mu^2 + \sigma^2).$$

(2.3)

$$E(Y|X) = E(Z{\cdot}X|X) = (\text{property of conditional expectation}) = X{\cdot}E(Z|X) = (\text{independence of } Z \text{ and } X) = X{\cdot}E(Z) = X{\cdot}\mu.$$

□

3 (3 points)

Are there two independent and identically distributed random variables X and Y such that $X - Y \sim U(-1, 1)$? Here $U(-1, 1)$ stands for uniform random variable on the interval $(-1, 1)$. If there are, construct X and Y explicitly and explain why $X - Y \sim U(-1, 1)$. If there are no such random variables, proof it. (Hint: Since this question is about all possible random variables, the concept of characteristic function might help.)

Solution. No, there are no such random variables! Here is a proof based on characteristic function.
If there were such X and Y such that $X - Y \sim U(-1,1)$, let $\varphi(t) = a + b\cdot i$ denote their characteristic function (for some real a and b), then characteristic function of $X - Y$ reads

$$\varphi_{X-Y}(t) = Ee^{it(X-Y)} = Ee^{itX}\cdot Ee^{-itY} = \varphi(t)\cdot\varphi(-t) = (a+b\cdot i)\cdot(a-b\cdot i) = a^2+b^2 \geq 0.$$

On the other hand, the characteristic function of $U(-1,1)$ is (from Appendix B):

$$\varphi_{U(-1,1)}(t) = \frac{\sin t}{t}.$$

Since $\varphi_{U(-1,1)}(t) = \frac{\sin t}{t}$ can be negative and positive, there is no way that $\varphi_{X-Y}(t) = \varphi_{U(-1,1)}(t)$ for all t.

□

4 (3 points)

Let $X_1 \sim Exp(1)$ and $X_2 \sim Exp(1)$ be independent exponential random variables, and $X_{(1)} \leq X_{(2)}$ be their order statistic. Show that $X_{(2)}$ and $X_1 + \frac{1}{2}X_2$ have the same distribution.

Solution. It is clear that the density function of $X_{(2)}$ is given as (see Book, Section 4.1)

$$f_{X_{(2)}}(x) = 2(1-e^{-x})e^{-x}.$$

Method-1: Note that the density of X_1 is $f_{X_1}(x_1) = e^{-x_1}$ for $x_1 \geq 0$, and the density of $\frac{1}{2}X_2$ is $f_{X_2}(x_2) = 2e^{-2x_1}$ for $x_2 \geq 0$. Then it is from convolution that

$$f_{X_1+\frac{1}{2}X_2}(x) = \int_{-\infty}^{\infty} f_{X_1}(y)f_{X_2}(x-y)dy = \int_0^x e^{-y}2e^{-2(x-y)}dy = 2(1-e^{-x})e^{-x} = f_{X_{(2)}}(x).$$

Method-2: We compute characteristic functions.

$$\varphi_{X_{(2)}}(t) = E(e^{itX_{(2)}}) = \int_0^\infty e^{itx}2(1-e^{-x})e^{-x} = 2\int_0^\infty e^{(it-1)x}dx - 2\int_0^\infty e^{(it-2)x}dx = \frac{-2}{it-1} + \frac{2}{it-2},$$
$$\varphi_{X_1+\frac{1}{2}X_2}(t) = E(e^{it(X_1+\frac{1}{2}X_2)}) = E(e^{itX_1})E(e^{i\frac{t}{2}X_2}) = \frac{1}{1-it}\cdot\frac{1}{1-i\frac{t}{2}}.$$

It is clear that $\varphi_{X_{(2)}}(t) = \varphi_{X_1+\frac{1}{2}X_2}(t)$.

□

5 (3 points)

Let $(X,Y)'$ be two dimensional random vector whose joint probability density function $f(x,y)$ is give as

$$f(x,y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2-2xy+2y^2)}, \qquad -\infty < x < \infty, -\infty < y < \infty.$$

(5.1) (1p) Is $(X,Y)'$ a two dimensional normal random vector? If not, explain why. If yes, find the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Lambda}$. (Hint: n-dimensional normal has density $f(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{n/2}\frac{1}{\sqrt{\det\boldsymbol{\Lambda}}}e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Lambda}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$ for $\mathbf{x}\in\mathbf{R}^n$.)
(5.2) (1p) Find the marginal probability density function $f_X(x)$ of X.
(5.3) (1p) Find the conditional expectation $E(X|Y=y)$.

Solution. (5.1) By comparing with n-dimensional density $f(\mathbf{x})$, we try to rewrite $f(x,y)$ in the following form

$$f(x,y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2-2xy+2y^2)} = \frac{1}{2\pi}e^{-\frac{1}{2}(x,y)\begin{pmatrix}a & c\\ c & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}},$$

where the matrix $\begin{pmatrix}a & c\\ c & b\end{pmatrix}$ is considered as $\boldsymbol{\Lambda}^{-1}$. It holds that

$$(x,y)\begin{pmatrix}a & c\\ c & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = ax^2+2cxy+bY^2 = \text{(should be)} = x^2-2xy+2y^2,$$

from which $a = 1, b = 2$ and $c = -1$, that is $\mathbf{\Lambda}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, which in turn implies $\mathbf{\Lambda} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Therefore,

$$(X,Y)' \sim N(\boldsymbol{\mu}, \mathbf{\Lambda}), \qquad \boldsymbol{\mu} = \mathbf{0}, \mathbf{\Lambda} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

(5.2). **Method-1:** It is directly from the solution of (5.1) that $X \sim N(0,2)$, therefore

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{x^2}{4}}.$$

Method-2: A direct integration gives

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2-2xy+2y^2)} dy = \ldots = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{x^2}{4}}.$$

(5.3) Just as in (5.2), the marginal density $f_Y(y)$ of Y can be obtained (note that $Y \sim N(0,1)$) as $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$.

$$\begin{aligned} E(X|Y=y) &= \int_{-\infty}^{\infty} x \cdot f_{X|Y=y}(x)dx = \int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{f_Y(y)} dx \\ &= \int_{-\infty}^{\infty} x \cdot \frac{\frac{1}{2\pi} e^{-\frac{1}{2}(x^2-2xy+2y^2)}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}} dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-y)^2} dx = y. \end{aligned}$$

□

6 (3 points)

Let $X_n \sim Bin(n, \frac{\lambda}{n})$ be Binomial with a constant parameter $\lambda > 0$. Prove that $X_n \xrightarrow{d} Po(\lambda)$ as $n \to \infty$, where $\xrightarrow{d}$ means convergence in distribution, and $Po(\lambda)$ stands for a Poisson random variable with parameter λ. (Hint: Transforms (Probability Generating Function (PGF), Moment Generating Function (MGF), and Characteristic Function (CF)) might help.)

Solution. PGF of X_n (Book, page 61) and PGF of $Po(\lambda)$ (Book, page 63) are:

$$g_{X_n}(t) = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n}t\right)^n = \left(1 + \frac{\lambda(t-1)}{n}\right)^n \to e^{\lambda(t-1)} = g_{Po(\lambda)}(t).$$

Then it is directly from the continuity theorem (Book, page 159) that $X_n \xrightarrow{d} Po(\lambda)$. Proof based on MGF or CF will be similar as above.

□

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\mathrm{Bin}(n,p)$, $n = 1, 2, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1 - qe^{it}}$
First success $\mathrm{Fs}(p)$, $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\mathrm{NBin}(n,p)$, $n = 1, 2, 3, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\frac{q}{p}$	$n\frac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m > 0$	$p(k) = e^{-m}\frac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it}-1)}$
Hypergeometric $H(N,n,p)$, $n = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $p = 0, \frac{1}{N}, \frac{2}{N}, \ldots, 1$	$p(k) = \frac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\operatorname{Var} X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$, $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a>0$	$f(x)=\frac{1}{a}e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\frac{1}{\Gamma(p)}x^{p-1}\frac{1}{a^p}e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\dots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}x^{\frac{1}{2}n-1}\left(\frac{1}{2}\right)^{n/2}e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}x^{r-1}(1-x)^{s-1}$, $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^{\beta}\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1) -\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2},\ n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2},$ $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2,$ $n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x)=\frac{1}{\pi}\cdot\frac{a}{a^2+(x-m)^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{imt-a\|t\|}$
$C(0,1)$	$f(x)=\frac{1}{\pi}\cdot\frac{1}{1+x^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{-\|t\|}$
Pareto	$f(x)=\frac{\alpha k^\alpha}{x^{\alpha+1}},\ x>k$	$\frac{\alpha k}{\alpha-1},\ \alpha>1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha>2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k>0,\ \alpha>0$				