

Examiner: Xiangfeng Yang (013-285788). **Things allowed:** a calculator, a self-written A4 paper (two sides).

Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.

Notation: 'A random variable X is distributed as...' is written as ' $X \in \dots$ or $X \sim \dots$ '

1 (3 points)

Let X be a continuous random variable with a probability density function $f_X(x) = \frac{1}{2}e^{-|x|}$ for $-\infty < x < \infty$. Define $Y = X^2$. Find the density function $f_Y(y)$ of Y .

2 (3 points)

Let $Y = Z \cdot X$ where $Z \sim N(\mu, \sigma^2)$ (normal) and $X \sim Be(p)$ (Bernoulli) are two independent random variables.

(2.1) (1p) Find the expectation $E(Y)$.

(2.2) (1p) Find the expectation $E(X \cdot Y \cdot Z)$.

(2.3) (1p) Find the conditional expectation $E(Y|X)$.

3 (3 points)

Are there two independent and identically distributed random variables X and Y such that $X - Y \sim U(-1, 1)$? Here $U(-1, 1)$ stands for uniform random variable on the interval $(-1, 1)$. If there are, construct X and Y explicitly and explain why $X - Y \sim U(-1, 1)$. If there are no such random variables, proof it. (Hint: Since this question is about all possible random variables, the concept of characteristic function might help.)

4 (3 points)

Let $X_1 \sim Exp(1)$ and $X_2 \sim Exp(1)$ be independent exponential random variables, and $X_{(1)} \leq X_{(2)}$ be their order statistic. Show that $X_{(2)}$ and $X_1 + \frac{1}{2}X_2$ have the same distribution.

5 (3 points)

Let $(X, Y)'$ be two dimensional random vector whose joint probability density function $f(x, y)$ is give as

$$f(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2 - 2xy + 2y^2)}, \quad -\infty < x < \infty, -\infty < y < \infty.$$

(5.1) (1p) Is $(X, Y)'$ a two dimensional normal random vector? If not, explain why. If yes, find the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Lambda}$. (Hint: n -dimensional normal has density $f(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{\sqrt{\det \boldsymbol{\Lambda}}} e^{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Lambda}^{-1}(\mathbf{x} - \boldsymbol{\mu})}$ for $\mathbf{x} \in \mathbf{R}^n$.)

(5.2) (1p) Find the marginal probability density function $f_X(x)$ of X .

(5.3) (1p) Find the conditional expectation $E(X|Y = y)$.

6 (3 points)

Let $X_n \sim Bin(n, \frac{\lambda}{n})$ be Binomial with a constant parameter $\lambda > 0$. Prove that $X_n \xrightarrow{d} Po(\lambda)$ as $n \rightarrow \infty$, where \xrightarrow{d} means convergence in distribution, and $Po(\lambda)$ stands for a Poisson random variable with parameter λ . (Hint: Transforms (Probability Generating Function (PGF), Moment Generating Function (MGF), and Characteristic Function (CF)) might help.)

Discrete Distributions

Following is a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions.

An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E X$	$\text{Var } X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\text{Be}(p)$, $0 \leq p \leq 1$	$p(0) = q$, $p(1) = p$; $q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\text{Bin}(n, p)$, $n = 1, 2, \dots$, $0 \leq p \leq 1$	$p(k) = \binom{n}{k} p^k q^{n-k}$, $k = 0, 1, \dots, n$; $q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\text{Ge}(p)$, $0 \leq p \leq 1$	$p(k) = pq^k$, $k = 0, 1, 2, \dots$; $q = 1 - p$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1 - qe^{it}}$
First success $\text{Fs}(p)$, $0 \leq p \leq 1$	$p(k) = pq^{k-1}$, $k = 1, 2, \dots$; $q = 1 - p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\text{NBin}(n, p)$, $n = 1, 2, 3, \dots$, $0 \leq p \leq 1$	$p(k) = \binom{n+k-1}{k} p^n q^k$, $k = 0, 1, 2, \dots$; $q = 1 - p$	$n \frac{q}{p}$	$n \frac{q}{p^2}$	$\left(\frac{p}{1 - qe^{it}}\right)^n$
Poisson $\text{Po}(m)$, $m > 0$	$p(k) = e^{-m} \frac{m^k}{k!}$, $k = 0, 1, 2, \dots$	m	m	$e^{m(e^{it} - 1)}$
Hypergeometric $H(N, n, p)$, $n = 0, 1, \dots, N$, $N = 1, 2, \dots$, $p = 0, \frac{1}{N}, \frac{2}{N}, \dots, 1$	$p(k) = \frac{\binom{Np}{k} \binom{Nq}{n-k}}{\binom{N}{n}}, \quad k = 0, 1, \dots, Np;$ $q = 1 - p;$ $n - k = 0, \dots, Nq$	np	$npq \frac{N-n}{N-1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	EX	$\text{Var } X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a, b)$	$f(x) = \frac{1}{b-a}, a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0, 1)$	$f(x) = 1, 0 < x < 1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1, 1)$	$f(x) = \frac{1}{2}, x < 1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\text{Tri}(a, b)$	$f(x) = \frac{2}{b-a} \left(1 - \frac{2}{b-a} \left x - \frac{a+b}{2} \right \right)$ $a < x < b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)} \right)^2$
Tri(-1, 1)				
	$f(x) = 1 - x , x < 1$	0	$\frac{1}{6}$	$\left(\frac{\sin \frac{t}{2}}{\frac{t}{2}} \right)^2$
Exponential				
$\text{Exp}(a), a > 0$	$f(x) = \frac{1}{a} e^{-x/a}, x > 0$	a	a^2	$\frac{1}{1-ait}$
Gamma				
$\Gamma(p, a), a > 0, p > 0$	$f(x) = \frac{1}{\Gamma(p)} x^{p-1} \frac{1}{a^p} e^{-x/a}, x > 0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square				
$\chi^2(n), n = 1, 2, 3, \dots$	$f(x) = \frac{1}{\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \left(\frac{1}{2} \right)^{n/2} e^{-x/2}, x > 0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace				
$L(a), a > 0$	$f(x) = \frac{1}{2a} e^{- x /a}, -\infty < x < \infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta				
$\beta(r, s), r, s > 0$	$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1} (1-x)^{s-1},$ $0 < x < 1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E X$	$\text{Var } X$	$\varphi_X(t)$
Weibull $W(\alpha, \beta), \alpha, \beta > 0$	$f(x) = \frac{1}{\alpha\beta} x^{(1/\beta)-1} e^{-x^{1/\beta}/\alpha}, x > 0$	$\alpha^\beta \Gamma(\beta + 1)$	$\alpha^{2\beta} (\Gamma(2\beta + 1) - \Gamma(\beta + 1)^2)$	*
Rayleigh $\text{Ra}(\alpha), \alpha > 0$	$f(x) = \frac{2}{\alpha} x e^{-x^2/\alpha}, x > 0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1 - \frac{1}{4}\pi)$	*
Normal $N(\mu, \sigma^2),$ $-\infty < \mu < \infty, \sigma > 0$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty < x < \infty$	μ	σ^2	$e^{i\mu t - \frac{1}{2}t^2\sigma^2}$
$N(0, 1)$	$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < x < \infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu, \sigma^2),$ $-\infty < \mu < \infty, \sigma > 0$	$f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{1}{2}(\log x - \mu)^2/\sigma^2}, x > 0$	$e^{\mu + \frac{1}{2}\sigma^2}$	$e^{2\mu}(e^{2\sigma^2} - e^{\sigma^2})$	*
(Student's) t $t(n), n = 1, 2, \dots$	$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n} \Gamma(\frac{n}{2})} \cdot d \frac{1}{(1 + \frac{x^2}{n})^{(n+1)/2}},$ $-\infty < x < \infty$	0	$\frac{n}{n-2}, n > 2$	*
(Fisher's) F $F(m, n), m, n = 1, 2, \dots$	$f(x) = \frac{\Gamma(\frac{m+n}{2}) (\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2}) \Gamma(\frac{n}{2})} \cdot \frac{x^{m/2-1}}{(1 + \frac{mx}{n})^{(m+n)/2}},$ $x > 0$	$\frac{n}{n-2},$ $n > 2$	$\frac{n^2(m+2)}{m(n-2)(n-4)} - \left(\frac{n}{n-2}\right)^2,$ $n > 4$	*

Continuous Distributions (continued)

Distribution, notation	Density	EX	$\text{Var } X$	$\varphi_X(t)$
Cauchy $C(m, a)$	$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2 + (x-m)^2}, \quad -\infty < x < \infty$	\bar{A}	\bar{A}	$e^{imt-a t }$
$C(0, 1)$	$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad -\infty < x < \infty$	\bar{A}	\bar{A}	$e^{- t }$
Pareto $\text{Pa}(k, \alpha), k > 0, \alpha > 0$	$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}}, \quad x > k$	$\frac{\alpha k}{\alpha-1}, \alpha > 1$	$\frac{\alpha k^2}{(\alpha-2)(\alpha-1)^2}, \alpha > 2,$	*