TAMS46: Probability Theory (Second Course) | Provkod: TEN1 | 30 October 2025, 14:00-18:00

Examiner: Xiangfeng Yang (013-285788). **Things allowed**: a calculator, a self-written/printed A4 paper (two sides).
Scores rating (Betygsgränser): 8-11 points giving rate 3; 11.5-14.5 points giving rate 4; 15-18 points giving rate 5.
Notation: 'A random variable X is distributed as...' is written as '$X \in$ ... or $X \sim$ ... '
Remark: Write down all necessary steps in your solutions in order to receive as many points as possible.

1 (3 points)

Let $X \sim Exp(1)$ and $Y \sim Exp(1)$ be independent exponential random variables.
(1.1) (1p) Find the probability density function $f_{X+Y}(u)$ of $X+Y$.
(1.2) (2p) Find the probability density function $f_{X-Y}(v)$ of $X-Y$.

Solution. Set $U = X - Y$ and $V = X + Y$. Then it follows that

$$X = \frac{U+V}{2}, \quad Y = \frac{V-U}{2}, \quad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}, \quad v > 0, -v < u < v.$$

Since $f_{X,Y}(x,y) = e^{-x-y}$ for $x > 0$ and $y > 0$, it follows that

$$f_{U,V}(u,v) = e^{-\frac{u+v}{2} - \frac{v-u}{2}} \cdot |J| = e^{-v} \cdot \frac{1}{2}, \quad v > 0, -v < u < v.$$

(1.1)

$$f_V(v) = \int_{-v}^{v} f_{U,V}(u,v)du = \int_{-v}^{v} e^{-v} \cdot \frac{1}{2} du = e^{-v} \cdot v, \quad v > 0.$$

(1.2) It is clear that $-v < u < v$ is equivalent to $|u| < v$, therefore,

$$f_U(u) = \int_{|u|}^{\infty} f_{U,V}(u,v)dv = \int_{|u|}^{\infty} e^{-v} \cdot \frac{1}{2} dv = e^{-|u|} \cdot \frac{1}{2}, \quad -\infty < u < \infty.$$

□

2 (3 points)

A stick of length 1 is cut off at a random point uniformly on $(0, 1)$, so the length X of the remaining piece is a uniform random variable $X \sim U(0, 1)$. This remaining piece is cut off again at a random point uniformly on $(0, X)$, then the length Y of the second remaining piece is a random variable $Y \sim U(0, X)$ (equivalently, it can be written as $Y|X = x \sim U(0, x)$ for $0 < x < 1$).
(2.1) (1p) Find the expectation $E(Y)$ of Y.
(2.2) (1p) Find the variance $V(Y)$ of Y.
(2.3) (1p) Find the probability $P(Y \leq y)$.

Solution. (2.1)

$$E(Y) = E(E(Y|X)) = E\left(\frac{X}{2}\right) = \frac{1}{2}E(X) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

(2.2)

$$V(Y) = E\left(V(Y|X)\right) + V\left(E(Y|X)\right) = E\left(\frac{X^2}{12}\right) + V\left(\frac{X}{2}\right) = \frac{1}{36} + \frac{1}{48} = \frac{7}{144} = 0.0486.$$

(2.3) For $0 < y < 1$ and set $g(u) = 1_{\{u \leq y\}}$,

$$\begin{aligned} P(Y \leq y) = E(g(Y)) = E(E(g(Y)|X)) &= \int_0^1 E(g(Y)|X = x) \cdot f_X(x)dx \\ &= \int_0^1 \left[\int_0^x g(u) f_{Y|X=x}(u)du\right] \cdot f_X(x)dx = \int_0^1 \left[\int_0^x 1_{\{u \leq y\}} f_{Y|X=x}(u)du\right] \cdot f_X(x)dx \\ &= \int_0^1 \left[\int_0^{\min(x,y)} \frac{1}{x} du\right] \cdot 1dx = \int_0^1 \frac{1}{x} \cdot \min(x,y)dx \\ &= \int_0^y \frac{1}{x} \cdot xdx + \int_y^1 \frac{1}{x} \cdot ydx = y - y\ln y. \end{aligned}$$

$\square$

3 (3 points)

Let $(X,Y)'$ be two dimensional random vector whose joint probability density function $f(x,y)$ is give as

$$f(x,y) = \frac{2}{5}\cdot(2x+3y), \qquad \text{if } 0<x<1 \text{ and } 0<y<1.$$

Find the conditional expectations $E(Y|X=x)$ and $E(X|Y=y)$.

Solution. The marginal density functions can be found as follows:

$$f_X(x) = \int_0^1 f(x,y)dy = \frac{4}{5}x + \frac{3}{5}, \text{ for } 0<x<1, \quad f_Y(y) = \int_0^1 f(x,y)dx = \frac{6}{5}y + \frac{2}{5}, \text{ for } 0<y<1.$$

Therefore,

$$E(Y|X=x) = \int_0^1 y f_{Y|X=x}(y)dy = \int_0^1 y\frac{f(x,y)}{f_X(x)}dy = \frac{2x+2}{4x+3}, \quad 0<x<1.$$
$$E(X|Y=y) = \int_0^1 x f_{X|Y=y}(x)dx = \int_0^1 x\frac{f(x,y)}{f_Y(y)}dx = \frac{9y+4}{18y+6}, \quad 0<y<1.$$

$\square$

4 (3 points)

Let $X_1 \sim Exp(1)$ and $X_2 \sim Exp(1)$ be independent exponential random variables, and $X_{(1)} \le X_{(2)}$ be their order statistic.
(4.1) (1p) Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent, and determine their distributions.
(4.2) (2p) Find the conditional expectations $E(X_{(2)}|X_{(1)}=x)$ and $E(X_{(1)}|X_{(2)}=y)$.

Solution. It is from Book (p.110. Theorem 3.1) that the joint density function of $(X_{(1)}, X_{(2)})$ is

$$f_{X_{(1)},X_{(2)}}(x,y) = 2f(x)f(y) = 2e^{-x-y}, \quad 0<x<y.$$

(4.1) Let $U = X_{(1)}$ and $V = X_{(2)} - X_{(1)}$. It follows that

$$X_{(1)} = U, \quad X_{(2)} = U+V, \quad J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \quad u>0, v>0.$$

Therefore, the joint density function of (U,V) is

$$\begin{aligned} f_{U,V}(u,v) &= 2e^{-u-(u+v)}\cdot|J| = 2e^{-2u}\cdot e^{-v}, \quad u>0, v>0, \\ &= f_U(u)\cdot f_V(v). \end{aligned}$$

Now it follows that U and V are independent. Furthermore, $U \sim Exp(1/2)$ and $V \sim Exp(1)$.
(4.2)

$$\begin{aligned} E(X_{(2)}|X_{(1)}=x) &= E(X_{(2)} - X_{(1)} + X_{(1)}|X_{(1)}=x) = E(X_{(2)} - X_{(1)}|X_{(1)}=x) + E(X_{(1)}|X_{(1)}=x) \\ &= (\text{independence of } U \text{ and } V) \quad E(X_{(2)} - X_{(1)}) + x = 1 + x. \end{aligned}$$

With (see Book (p.102)) $f_{X_{(2)}}(y) = 2(1-e^{-y})e^{-y}$,

$$E(X_{(1)}|X_{(2)}=y) = \int_0^y x\cdot f_{X_{(1)}|X_{(2)}=y}(x)dx = \int_0^y x\cdot\frac{f_{X_{(1)},X_{(2)}}(x,y)}{f_{X_{(2)}}(y)}dx = 1 + \frac{y}{1-e^y}.$$

$\square$

5 (3 points)

Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables with a common characteristic function $\varphi_X(t) = \frac{1}{2} + \frac{1}{4}(e^{-it} + e^{it})$ for $t \in \mathbb{R}$. Let $N \sim Po(\lambda)$ be a Poisson random variable which is independent of $X_1, X_2, \ldots$. Set

$$Z = X_1 + X_2 + \ldots + X_N.$$

(5.1) (1p) Find the expectation $E(X)$ of X.
(5.2) (1p) Find the expectation $E(Z)$ of Z.
(5.3) (1p) Find the variance $V(Z)$ of Z.

Solution. (5.1) It is from Book (p.74, Theorem 4.7) that $\varphi^{(k)}(0) = i^k E(X^k)$, therefore

$$E(X) = \varphi'(0)/i = \frac{1}{4}(-ie^{-i0} + ie^{i0})/i = 0.$$

(5.2) It is from Book (p.81, Theorem 6.2) that

$$E(Z) = E(X) \cdot E(N) = 0 \cdot \lambda = 0.$$

(5.3) It is from Book (p.81, Theorem 6.2) that

$$V(Z) = E(N) \cdot V(X) + (E(X))^2 \cdot V(N) = \lambda \cdot \frac{1}{2} + 0 \cdot \lambda = \frac{\lambda}{2},$$

where $V(X)$ is computed as follows

$$V(X) = E(X^2) - (E(X))^2 = E(X^2) = -\varphi''(0) = \frac{1}{2}.$$

□

6 (3 points)

Let $X_1, X_2, \ldots$ and X be one dimensional normal random variables.
(6.1) (1p) If $\lim_{n\to\infty} E(X_n) = E(X)$ and $\lim_{n\to\infty} V(X_n) = V(X)$, is it true $X_n \xrightarrow{d} X$ as $n \to \infty$? where $\xrightarrow{d}$ means convergence in distribution. If it is true, then prove it. If it is not true, then construct a counterexample.
(6.2) (1p) If $\lim_{n\to\infty} E(X_n) = E(X)$ and $\lim_{n\to\infty} V(X_n) = V(X)$, is it true $X_n^k \xrightarrow{d} X^k$ as $n \to \infty$ for any fixed positive integer k? If it is true, then prove it. If it is not true, then construct a counterexample.
(6.3) (1p) If $X_n \xrightarrow{d} X$ as $n \to \infty$, is it true that $\lim_{n\to\infty} E(X_n) = E(X)$ and $\lim_{n\to\infty} V(X_n) = V(X)$? If it is true, then prove it. If it is not true, then construct a counterexample.

Solution. Let $X_n \sim N(\mu_n, \sigma_n^2)$ and $X \sim N(\mu, \sigma^2)$. The characteristic functions are

$$\varphi_{X_n}(t) = e^{i\mu_n t - \frac{1}{2}t^2\sigma_n^2}, \qquad \varphi_X(t) = e^{i\mu t - \frac{1}{2}t^2\sigma^2}.$$

(6.1) If $\mu_n \to \mu$ and $\sigma_n \to \sigma$, then it is clear that $\varphi_{X_n}(t) \to \varphi_X(t)$. This implies that $X_n \xrightarrow{d} X$.

(6.2) The characteristic functions of X_n^k and X^k are

$$\varphi_{X_n^k}(t) = E(e^{itX_n^k}) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}\sigma_n} e^{-(x-\mu_n)^2/(2\sigma_n^2)} dx$$
$$\varphi_{X^k}(t) = E(e^{itX^k}) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} dx.$$

If $\mu_n \to \mu$ and $\sigma_n \to \sigma$, then it is from the dominated convergence theorem that $\varphi_{X_n^k}(t) \to \varphi_{X^k}(t)$. This implies that $X_n^k \xrightarrow{d} X^k$.

(6.3) It is from Book (p.159, Remark 4.2) that if $X_n \xrightarrow{d} X$, then $\varphi_{X_n}(t) \to \varphi_X(t)$. That is

$$e^{i\mu_n t - \frac{1}{2}t^2\sigma_n^2} = e^{-\frac{1}{2}t^2\sigma_n^2}(\cos(\mu_n t) + i\sin(\mu_n t)) \to e^{-\frac{1}{2}t^2\sigma^2}(\cos(\mu t) + i\sin(\mu t))$$

for all $-\infty < t < \infty$. This implies that $\mu_n \to \mu$ and $\sigma_n \to \sigma$. □

Discrete Distributions

Followingis a list of discrete distributions, abbreviations, their probability functions, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Probability function	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
One point $\delta(a)$	$p(a) = 1$	a	0	e^{ita}
Symmetric Bernoulli	$p(-1) = p(1) = \frac{1}{2}$	0	1	$\cos t$
Bernoulli $\mathrm{Be}(p)$, $0 \le p \le 1$	$p(0) = q,\ p(1) = p;\ q = 1 - p$	p	pq	$q + pe^{it}$
Binomial $\mathrm{Bin}(n, p)$, $n = 1, 2, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1, \ldots, n;\ q = 1 - p$	np	npq	$(q + pe^{it})^n$
Geometric $\mathrm{Ge}(p)$, $0 \le p \le 1$	$p(k) = pq^k,\ k = 0, 1, 2, \ldots;\ q = 1 - p$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$	$\dfrac{p}{1 - qe^{it}}$
First success $\mathrm{Fs}(p)$, $0 \le p \le 1$	$p(k) = pq^{k-1},\ k = 1, 2, \ldots;\ q = 1 - p$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	$\dfrac{pe^{it}}{1 - qe^{it}}$
Negative binomial $\mathrm{NBin}(n, p)$, $n = 1, 2, 3, \ldots$, $0 \le p \le 1$	$p(k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \ldots;$ $q = 1 - p$	$n\dfrac{q}{p}$	$n\dfrac{q}{p^2}$	$(\frac{p}{1 - qe^{it}})^n$
Poisson $\mathrm{Po}(m)$, $m > 0$	$p(k) = e^{-m}\dfrac{m^k}{k!},\ k = 0, 1, 2, \ldots$	m	m	$e^{m(e^{it} - 1)}$
Hypergeometric $H(N, n, p)$, $n = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $p = 0, \frac{1}{N}, \frac{2}{N}, \ldots, 1$	$p(k) = \dfrac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}},\quad k = 0, 1, \ldots, Np;$ $q = 1 - p;$ $n - k = 0, \ldots, Nq$	np	$npq\dfrac{N - n}{N - 1}$	*

Continuous Distributions

Following is a list of some continuous distributions, abbreviations, their densities, means, variances, and characteristic functions. An asterisk (*) indicates that the expression is too complicated to present here; in some cases a closed formula does not even exist.

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Uniform/Rectangular				
$U(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
$U(0,1)$	$f(x)=1,\ 0<x<1$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{e^{it}-1}{it}$
$U(-1,1)$	$f(x)=\frac{1}{2},\ \lvert x\rvert<1$	0	$\frac{1}{3}$	$\frac{\sin t}{t}$
Triangular				
$\mathrm{Tri}(a,b)$	$f(x)=\frac{2}{b-a}\left(1-\frac{2}{b-a}\left\lvert x-\frac{a+b}{2}\right\rvert\right)$ $a<x<b$	$\frac{1}{2}(a+b)$	$\frac{1}{24}(b-a)^2$	$\left(\frac{e^{itb/2}-e^{ita/2}}{\frac{1}{2}it(b-a)}\right)^2$
$\mathrm{Tri}(-1,1)$	$f(x)=1-\lvert x\rvert,\ \lvert x\rvert<1$	0	$\frac{1}{6}$	$\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$
Exponential $\mathrm{Exp}(a),\ a>0$	$f(x)=\frac{1}{a}\,e^{-x/a},\ x>0$	a	a^2	$\frac{1}{1-ait}$
Gamma $\Gamma(p,a),\ a>0,\ p>0$	$f(x)=\frac{1}{\Gamma(p)}\,x^{p-1}\,\frac{1}{a^p}\,e^{-x/a},\ x>0$	pa	pa^2	$\frac{1}{(1-ait)^p}$
Chi-square $\chi^2(n),\ n=1,2,3,\ldots$	$f(x)=\frac{1}{\Gamma(\frac{n}{2})}\,x^{\frac{1}{2}n-1}\,\left(\frac{1}{2}\right)^{n/2}\,e^{-x/2},\ x>0$	n	$2n$	$\frac{1}{(1-2it)^{n/2}}$
Laplace $L(a),\ a>0$	$f(x)=\frac{1}{2a}e^{-\lvert x\rvert/a},\ -\infty<x<\infty$	0	$2a^2$	$\frac{1}{1+a^2t^2}$
Beta $\beta(r,s),\ r,s>0$	$f(x)=\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\,x^{r-1}(1-x)^{s-1},$ $0<x<1$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Weibull $W(\alpha,\beta)$, $\alpha,\beta>0$	$f(x)=\frac{1}{\alpha\beta}x^{(1/\beta)-1}\,e^{-x^{1/\beta}/\alpha},\ x>0$	$\alpha^\beta\,\Gamma(\beta+1)$	$a^{2\beta}\big(\Gamma(2\beta+1)$ $-\Gamma(\beta+1)^2\big)$	*
Rayleigh $\mathrm{Ra}\,(\alpha)$, $\alpha>0$	$f(x)=\frac{2}{\alpha}\,x\,e^{-x^2/\alpha},\ x>0$	$\frac{1}{2}\sqrt{\pi\alpha}$	$\alpha(1-\frac{1}{4}\pi)$	*
Normal $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $\sigma>0$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2},$ $-\infty<x<\infty$	μ	σ^2	$e^{i\mu t-\frac{1}{2}t^2\sigma^2}$
$N(0,1)$	$f(x)=\dfrac{1}{\sqrt{2\pi}}\,e^{-x^2/2},\ -\infty<x<\infty$	0	1	$e^{-t^2/2}$
Log-normal $LN(\mu,\sigma^2)$, $-\infty<\mu<\infty,\ \sigma>0$	$f(x)=\dfrac{1}{\sigma x\sqrt{2\pi}}\,e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2},\ x>0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu}\big(e^{2\sigma^2}-e^{\sigma^2}\big)$	*
(Student's) t $t(n)$, $n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}\cdot d\frac{1}{(1+\frac{x^2}{n})^{(n+1)/2}},$ $-\infty<x<\infty$	0	$\dfrac{n}{n-2}$, $n>2$	*
(Fisher's) F $F(m,n)$, $m,n=1,2,\ldots$	$f(x)=\frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\cdot\frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}},$ $x>0$	$\frac{n}{n-2}$, $n>2$	$\frac{n^2(m+2)}{m(n-2)(n-4)}-\Big(\frac{n}{n-2}\Big)^2,$ $n>4$	*

Continuous Distributions (continued)

Distribution, notation	Density	$E\,X$	$\mathrm{Var}\,X$	$\varphi_X(t)$
Cauchy				
$C(m,a)$	$f(x)=\frac{1}{\pi}\cdot\frac{a}{a^2+(x-m)^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{imt-a\lvert t\rvert}$
$C(0,1)$	$f(x)=\dfrac{1}{\pi}\cdot\dfrac{1}{1+x^2},\ -\infty<x<\infty$	$\nexists$	$\nexists$	$e^{-\lvert t\rvert}$
Pareto	$f(x)=\dfrac{\alpha k^\alpha}{x^{\alpha+1}},\ x>k$	$\dfrac{\alpha k}{\alpha-1},\ \alpha>1$	$\dfrac{\alpha k^2}{(\alpha-2)(\alpha-1)^2},\ \alpha>2,$	*
$\mathrm{Pa}\,(k,\alpha),\ k>0,\ \alpha>0$				