TANA09 Lecture 8

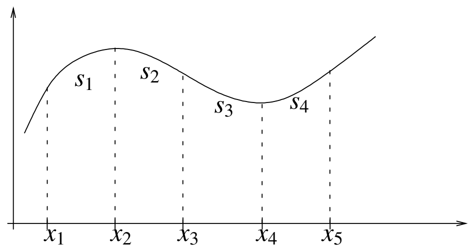
Approximating splines

- *B*-splines. Least squares fits.

Desiogn of curves and surfaces.

- Parametric curves. Sketching programs.
- Beziér curves. Design of curves and surfaces.
- Application - Mathematical description of fonts.

Cubic splines



Definition A function $s(x)$ is a *cubic spline* with *nodes* $x_1, \ldots, x_n$ if

1. $s(x)$, $s'(x)$, and $s''(x)$ are *continuous* on $[x_1, x_n]$.

2. $s(x)$ is a cubic polynomial on each intervall $[x_i, x_{i+1}]$.

Lemma The set of cubic splines $\mathcal{S}$, with nodes $x_1, \ldots, x_n$, is a *linear space* aof dimension $n+2$.

Find $n+2$ *linearly independent* cubic splines and we have a *basis* for the space.

How to pick a good set of basis functions?

Can also form spaces of linear or quadratic splines but it is hardly ever used.

B-Splines

Theorem Let $x_1, \ldots, x_n$ be given nodes. To each interval $[x_i, x_{i+4}]$ there is a cubic spline $B_i(x)$ with the properties

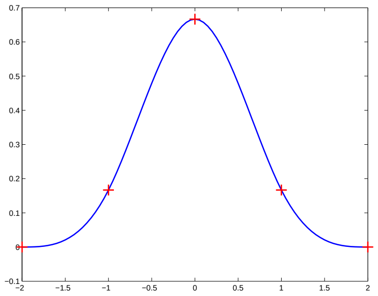
1. $B_i(x) = 0$ för $x \leq x_i$ och $x \geq x_{i+4}$.

2. $B_i(x_{i+1}) = B_i(x_{i+3}) = 1/6$ and $B_i(x_{i+2}) = 2/3$.

This function is called the cubic *B*-spline.

Theorem Let $x_{-2}, x_{-1}, \ldots, x_{n+2}, x_{n+3}$ be the nodes. The cubic *B*-splines $\{B_i(x)\}_{i=-2}^{n-1}$ form a *basis* for the space $\mathcal{S}$ and

$$\sum_{i=-2}^{n-1} B_i(x) = 1, \qquad x_1 < x < x_n.$$

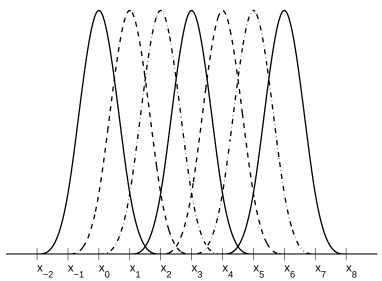


Exempel Let $B(x)$ be the *natural* cubic spline that interpolates the table

x	-2	-1	0	1	2
$B(x)$	0	1/6	2/3	1/6	0

If the nodes $x_{-2}, \ldots, x_{n+3}$ are *equidistant*, with step size $h = x_2 - x_1$, then

$$B_i(x) = B((x - x_i)/h).$$



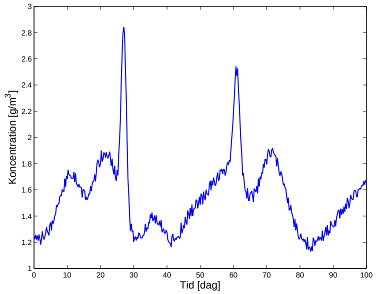
All B-splines needed for the interval $x_1 < x < x_5$. In total 7 basis functions are needed.

We have

$$\mathcal{S} = \text{span}(B_{-2}(x), B_{-1}(x), \ldots, B_4(x)).$$

Least squares fits with splines

Example Study the concentration of a chemical substance in a river. We measure the the concentration four times each day, during 100 days. The measurents contain noise we want to sliminate. We also want to compute integrals and derivatives of the concentration



Pick a *coarse grid* $x_{-2} < x_{-1} < \cdots < x_{N+3}$ such that att $x_1 = 0$ och $x_N = 100$. The number of grid points determine how much details the spline approximation will have.

Ansatz

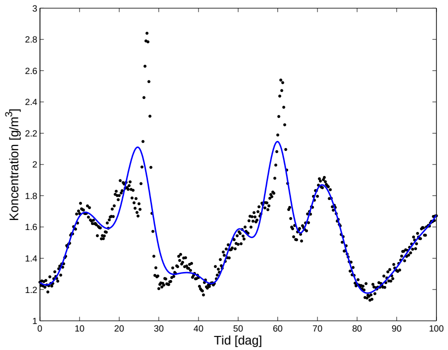
$$F(t) \approx s(x) = \sum_{i=-2}^{N-1} c_i B_i(x).$$

Also pick a *dense grid*

$$0 = x'_1 < x'_2 < \ldots < x'_1 < x'_n = 100,$$

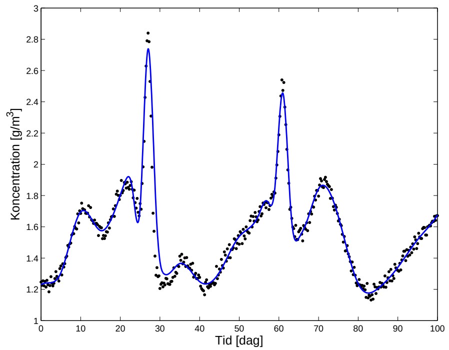
corresponding to the measurements, i.e. $F_i = F(x'_i)$ are known.

Find the coefficients c by solving a linear least squares problem $Ac = F$.



The result using a step size $h_1 = 5$ days, or 23 B-splines.

The approximation is good except for near the two spikes. Extend with a few extra basis functions and use $h_2 = 2$ near the spikes.

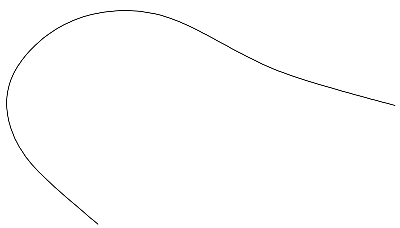


The approximation using a total of 29 basis functions. Near each spike we have 3 functions with $h_2 = 2$.

Remark Very flexible way to approximate a vector containing measurements by a differentiable function. Piecewise polynomials mean it is easy to perform analytic calculations.

A simple sketching program

Most sketching programs, e.g. `xfig`, has a function to draw curves by clicking on a number of points.



This is not the graph of a function $y = f(x)$. What to do instead?

Parametric curves

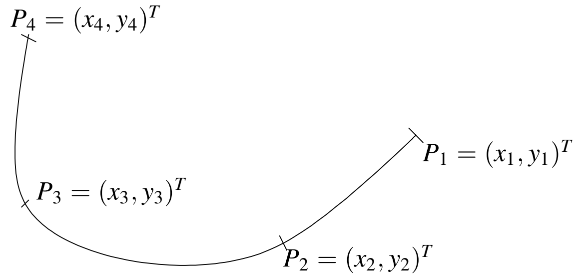
Definition A *parametric curve* in the plane is given by

$$s(t) = (x(t)\,,\, y(t))^T, \qquad a < t < b.$$

Each curve in the plane is described by *two* functions $x(t)$ and $y(t)$.

For a curve in $\mathbb{R}^3$ we also need a function $z(t)$.

Example We want a curve $s(t)$ to interpolate a number of given points.



The curve $s(t)$ should interpolate P_1, P_2, P_3, and P_4.

Chose a parameter interval $a = t_1 \leq t \leq t_4 = b$ and find *two* cubic splines that interpolates the tables

t	t_1	t_2	t_3	t_4
$x(t)$	x_1	x_2	x_3	x_4

and

t	t_1	t_2	t_3	t_4
$y(t)$	y_1	y_2	y_3	y_4

Then

$$s(t_i) = (x(t_i), y(t_i))^T = (x_i, y_i)^T = P_i.$$

The curve $s(t) = (x(t), t(t)^T$, $a \leq t \leq b$, interpolates the given points.

Example Find a curve that interpolates the points

t	0	1	3	5
$x(t)$	8.3	6.0	2.1	2.2
$y(t)$	4.8	2.8	3.7	7.3

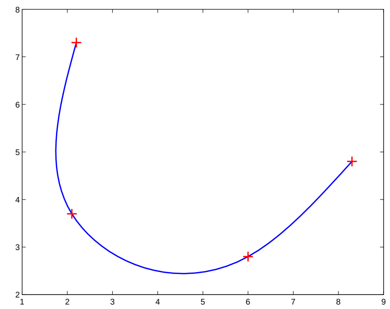
with natural end point conditions.

In Matlab we can use `ginput()` to click on points and create the table.

Given vectors t, x and y we write

```
>> px = csape(t,x,'variational');
>> py = csape(t,y,'variational');
>> tt=0:0.1:5;
>> plot( ppval(px,tt) , ppval(py,tt),'b-');
>> plot( x,y,'r+');
```

The result is a curve that interpolates the table



Effects of the end points conditions

Definition The *tangent* at the point $s(t)$ is

$$s'(t) = (x'(t), y'(t))^T.$$

The tangent of a curve shows the *direction* at a certain point.

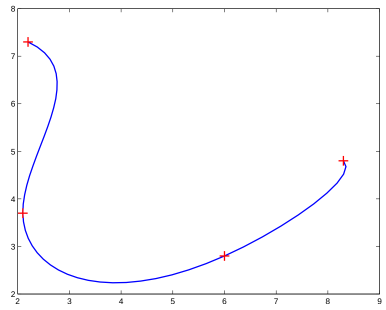
Example Use the same table but chose

$$s'(0) = (1, -1)^T, \quad \text{and} \quad s'(5) = (-2, 1)^T.$$

Whatr happens+

Given vectors t, x and y we instead write

```
>> px = csape(t,x,'complete',[1 -2]);
>> py = csape(t,y,'complete',[-1 1]);
>> tt=0:0.1:5;
>> plot( ppval(px,tt) , ppval(py,tt),'b-');
>> plot( x,y,'r+');
```

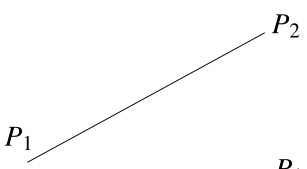


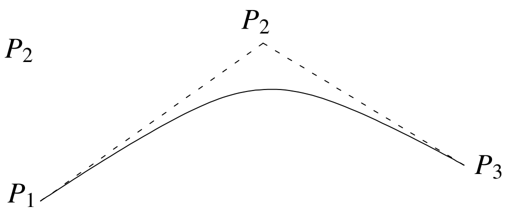
The tangent of the curve is altered at the end points.

Beziér curves

Definition A Beziér curve (or surface) is determined by a number of *control points*.

Example



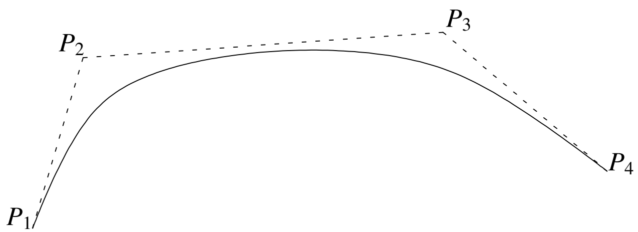


A *linear* Beziér curve is determined by two points. A *quadratic* curve is given by three points.

Beziér curves are *approximating* splines.

Cubic Beziér curves

A *cubic Beziér curve* is given by four control points.



The curve is given by

$$s(t) = (1-t)^3 P_1 + 3(1-t)^2 t P_2 + 3(1-t)t^2 P_3 + t^3 P_4.$$

Lemma A *cubic Beziér curve* $s(x)$ interpolates the first and last control point, i.e. $s(0) = P_1$ and $s(1) = P_4$.

Definition The *convex hull* formed by the points $\{P_i\}_{i=1}^n$ consists of all points that can be written as a convex linear coombination of $\{P_i\}_{i=1}^n$.

Example What is the convex hull of the points

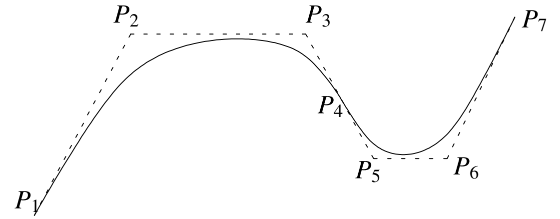
$$P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{and } P_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Theorem A Beziér curve is located within the convex hull formed by its control points.

Theorem A Beziér curve with control points $\{P_i\}_{i=1}^n$ has a tangent, at the start and end, given by

$$s'(0) = \alpha(P_2 - P_1), \text{ and } s'(1) = \alpha(P_n - P_{n-1}), \quad \alpha > 0.$$

We can easily control the slope of the curve at the end points.



Two cubic Beziér curves given by control points $\{P_1, P_2, P_3, P_4\}$ and $\{P_4, P_5, P_6, P_7\}$. The interpolation point P_4 is *common*.

The condition $P_4 - P_3 = P_5 - P_4$ gives a continuous tangent direction.

Vectorized fonts

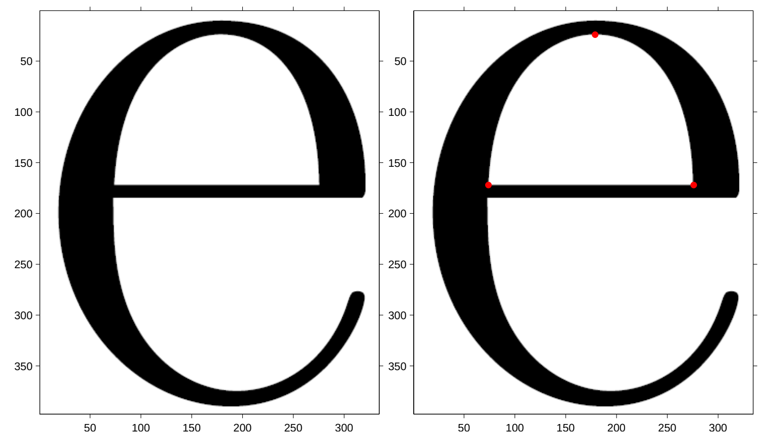
A document can be printed on different printers with various resolutions,. We need to be able to generate bitmap images representing characters with different resolutions.

Example Three letters from *Computer Modern*

Sei

How to define the shape of the letters?

Example To represent the letter *e* we introduce a number of control points.



First pick a few *interpolation points* anbd then a few extra *control points*.

First pick interpolation points and plot a line segment.

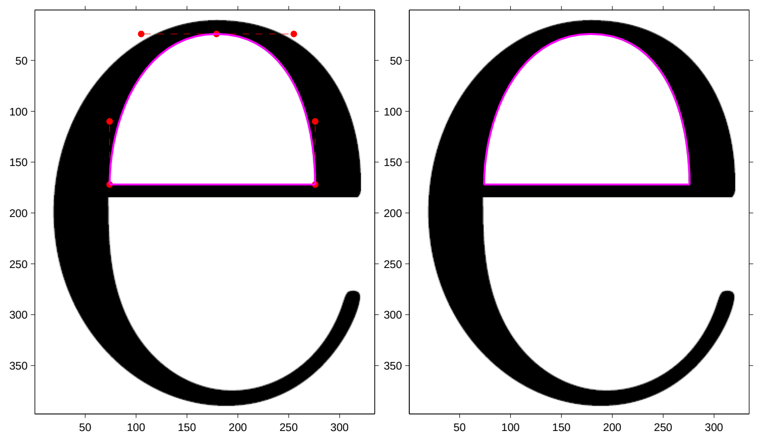
```
>>  P=[ 74  276 179
       172  172  24];
>> plot( P(1,1:2),P(1,1:2)
```

Pick a number of control points

```
>> C = [ 74   105   255   276]
         110    24    24   110];
```

and draw a dashed line from interpolation point P_1 and control point C_1 by

```
>> plot([P(1,1),C(1,1)],[P(2,1),C(2,1)],'r--')
```

The end result with or without control points visible.

In total 2 line segments and 11 Beziér curves are needed to describe the letter. Font packages are small.

Beziér surfaces

Introduce interpolation pointsP_1, P_2 and P_3 and alos control points C_1, C_2 and C_3.

A quadratic surface is given by

$$s(x,y) = P_1x^2 + P_2y^2 + P_3(1-x-y)^2 + C_1 2xy +$$

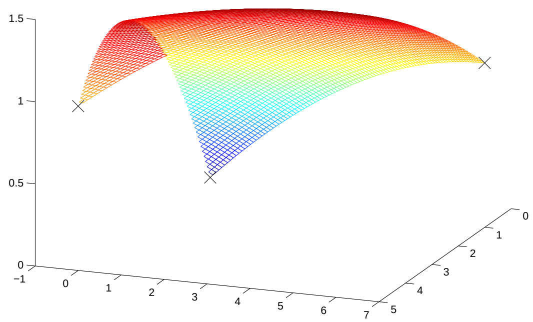
$$C_2 2x(1-x-y) + C_3 2y(1-x-y),$$

where $0 \leq x \leq 1, 0 < y \leq 1$ and $x + y \leq 1$.

Pick interpolation points and control points

```
P=[0 0 0 ; 0 5 1 ; 7 1 1 ]';
C=[0 3 2 ; 4 1 1 ; 5 4 2]';
```

Draw the surface.



The control points are higher, in the z direction, which gives the surface its curvature.

Summary

- The set of splines with given nodes is a linear space. The *B*-splines provide a good basis set. Can do least squares fits.
- Splines can be used to design parametric curves.
- Beziér curves or surfaces is the standard method to describe more complicated geometrical shapes.
- The definition of most vectorized fonts use Beziér curves.