Lecture 2

(1) P.3 <u>proof</u> Have to show that $Ax=b$ has a unique solution for all $b \in \mathbb{R}^n$. If x_1 and x_2 are two solutions then $A(x_1-x_2)=Ax_1-Ax_2=b-b=0$ so $x_1-x_2 \in \text{null}(A)=\{0\} \Rightarrow x_1=x_2$ so uniqueness.

Also: $\text{range}(A)=\mathbb{R}^n$ due to previous lemma.

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(2) P.3

$$A=\begin{pmatrix} 10^{-6} & & \\ & 10^{-6} & \\ & & 10^{-6} \end{pmatrix} \Rightarrow \det(A)=10^{-18} < \mu =$$ machine precision.

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(3) P.4 Row operation $(1\,2\,2\,|\,3)-0.25\,(4\,4\,2\,|\,6)$

$$=(0\;1\;1.5\;|\;1.5)$$

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(4) P.6

$$M^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix}$$

Add back the same multiple of the first row.

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(5) p.9

$$P_{23}M_1^{-1}P_{23}=P_{23}\begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix}P_{23}^T = P_{23}\begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 0 & 1 \\ m_{31} & 1 & 0 \end{pmatrix} \Rightarrow$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ m_{31} & 1 & 0 \\ m_{21} & 0 & 1 \end{pmatrix}=\hat{M}_1^{-1}$$

also

$$\hat{M}_1^{-1}M_2^{-1}=\hat{M}_1^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ m_{31} & 1 & 0 \\ m_{21} & m_{32} & 1 \end{pmatrix}$$

(6) If $PA = LU$ then

P.10 $\det(A) = \det(P^{-1})\det(L)\det(U) =$

$$= (\pm 1) \cdot 1 \cdot \prod_{i=1}^{n} u_{ii}$$

If Gaussian elimination works then A is non singular.

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(7) $PAx = Pb \Rightarrow LUx = Pb \Rightarrow$

P.10 $x = U^{-1}L^{-1}(Pb)$ " = " $U \backslash (L \backslash (P*b))$;

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(8) No pivoting means

$$A = LU = L(D\tilde{L}^T) = (LD^{1/2})(\tilde{L}D^{1/2})^T$$

$$= R^T\tilde{R}.$$

A symmetric $\Rightarrow R = \tilde{R}$

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(9) <u>proof</u> Have $A(x+\delta x) = b + \delta b \Rightarrow \delta x = A^{-1}\delta b$ and $\|\delta x\| \le \|A^{-1}\| \cdot \|\delta b\|$. Also $\|b\| \le \|A\| \cdot \|x\|$

Thus

$$\frac{\|\delta x\|}{\|x\|} \le \frac{\|A^{-1}\| \cdot \|\delta b\|}{\|b\| / \|A\|} = \underbrace{\|A\| \, \|A^{-1}\|}_{\kappa(A)} \cdot \frac{\|\delta b\|}{\|b\|}$$

Valid for $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$

(10) In Matlab $\mu \approx 1.1 \cdot 10^{-16}$ is the machine precision.

P.13 Best case. $\|r\| \approx 10^{-14} - 10^{-16}$

$$\kappa_2(A) \approx 10^{4} - 10^{12}$$

Relative error $10^{-12} - 10^{-2}$

Rough estimate of the error.

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(11) $\|Ax - b\|_2$ is minimized if $r \perp \text{Range}(A)$

P.14 or $(b - Ax)^T A = 0 \Leftrightarrow \boxed{A^T A x = A^T b}$

The proof is based on having a scalar product.

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(12) Modell: $p(t_k) = c_0 + c_1 t_k + c_2 t_k^2, \quad 1 \le k \le m$

P.17 or

$$\begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \\ 1 & t_m & t_m^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \qquad \boxed{m \times 3}$$

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(13) Let $Q = (q_1, q_2, \dots, q_m)$ then for $x \in \mathbb{R}^m$

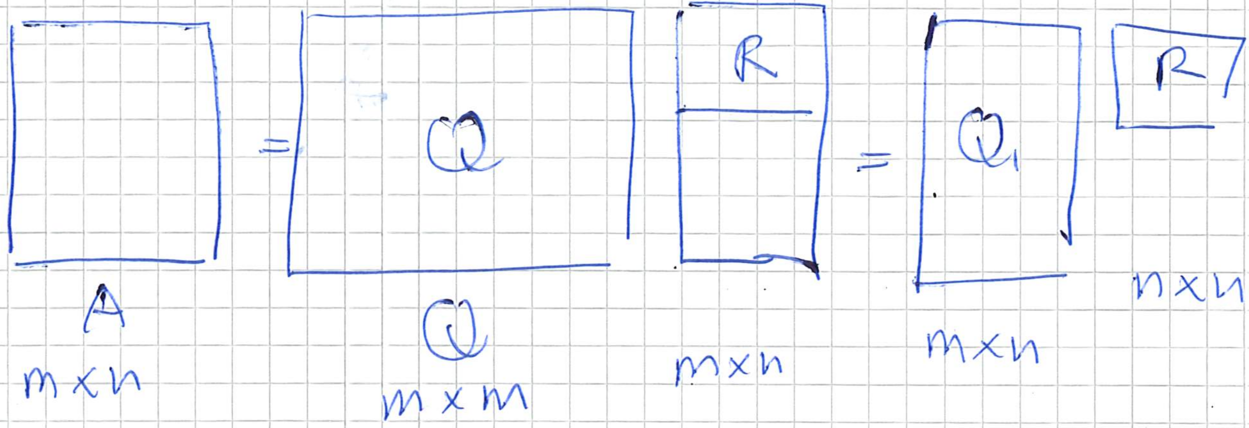
P.19

$$y = QQ^T x = Q \begin{pmatrix} q_1^T x \\ \vdots \\ q_m^T x \end{pmatrix} = (q_1^T x) q_1 + \dots + (q_m^T x) q_m$$

Orthogonal projection on $\text{span}(q_1, \dots, q_n)$

(14)

p. 21



To store Q requires m^2 memory slots.

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(15) $$\|Ax-b\|_2^2 = \left\|Q\begin{pmatrix} R \\ 0 \end{pmatrix}x - b\right\|_2^2 \underbrace{=}_{Q=(Q_1\ Q_2)} \left\|\begin{pmatrix} Rx - Q_1^T b \\ 0 - Q_2^T b \end{pmatrix}\right\|_2^2$$

$$= \|Rx - \underbrace{Q_1^T}_{\text{basis for Range}(A)} b\|_2^2 + \|\underbrace{Q_2^T}_{\text{basis for null}(A)} b\|_2^2$$

so minimum for $x = R^{-1}Q_1^T b$