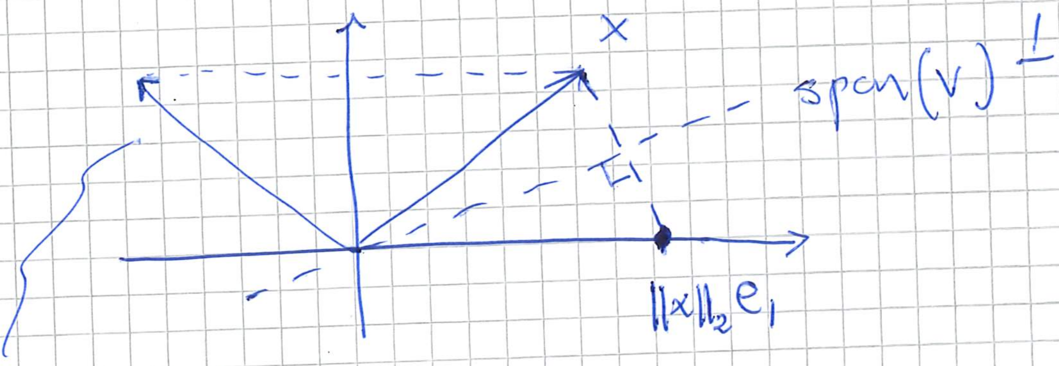
Lecture 3

p.3

(1) Find a H such that $Hx = \begin{pmatrix} \pm\|x\|_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Lemma $H = I - 2\frac{vv^T}{v^Tv}$ is a reflection in $\text{span}(v)^{\perp}$.



$v = x - \|x\|_2 e_1$

If $x = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \Rightarrow v = 0$ so pick $v = x + \text{sign}(x_1)\|x\|_2 e_1$

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(2) proof

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$$Hx = \left(I + \frac{2}{v^Tv} vv^T\right)x = x - \left(\frac{2}{v^Tv}\right)(v^Tx)v$$

needs $O(m)$ operations.

To get R we need n reflections. Each reflection should be applied to all columns of A

$$\Rightarrow O(n \cdot n \cdot m)$$

Q, Q_1 are obtained by applying reflections to I:

$$Q = \underbrace{\underbrace{H_1^T H_2^T \cdots H_n^T}_{n} \underbrace{I}_{m \text{ cols}}}_{O(nm^2)} \qquad Q_1 = \underbrace{\underbrace{H_1^T \cdots H_n^T}_{n} \overbrace{[e_1, e_2 \dots e_n]}^{n \text{ cols}}}_{O(n^2 m)}$$

(3) <u>proof</u> $\|Ax-b\|_2^2 = \|[A, b]\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2^2 =$

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$$= \|Q\underbrace{\begin{pmatrix} R \\ 0 \end{pmatrix}}_{(n+1)\times(n+1)}\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2^2 = \|\begin{pmatrix} R_1 & \gamma \\ 0 & r_{n+1} \end{pmatrix}\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2^2$$

$$= \|R_1 x - \gamma\|_2^2 + |r_{n+1}|^2$$

Solve $R_1 x = \gamma$. The minimum is $|r_{n+1}|$ //

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(4)

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$$\begin{pmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ \vdots & \vdots & \vdots \\ x_m^2+y_m^2 & x_m & y_m \end{pmatrix}\begin{pmatrix} a \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

A, $m\times 3$ $\quad x$, 3×1 $\quad b$, $m\times 1$

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(5) $A_1 = Q_1 R_1 \Rightarrow Q_1^T A_1 = R_1$ so

$$\begin{pmatrix} Q_1^T & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} A_1 \\ a_{14}^T \end{pmatrix}u - \begin{pmatrix} b_1 \\ b_{14} \end{pmatrix}\right) \text{"="} \begin{pmatrix} R_1 u - Q_1^T b \\ a_{14}u - b_{14} \end{pmatrix}$$

Need Q_2 to get the correct residual.

(6) A 4x4 Rotation G_{24} is

P.13

$$G_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & -s & 0 & c \end{pmatrix} \text{ or } G([2\ 4],[2\ 4]) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

Let $x = (x_1, x_2, x_3, \cancel{x_4})^T$ and apply G_{12} and G_{13}

$$G_{12} \begin{pmatrix} x \\ x \\ x \end{pmatrix} = \begin{pmatrix} + \\ 0 \\ x \end{pmatrix} \begin{array}{l} \leftarrow \text{changed non-zero} \\ \leftarrow \text{zero} \\ \leftarrow \text{unchanged non-zero} \end{array}$$

$$G_{13} \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} + \\ 0 \\ 0 \end{pmatrix} //$$

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(7) $G_{ij}A \Leftrightarrow A([i,j],:) \in \mathbb{R}^{2 \times n}$ is rotated. n 2×2 rotations $\Rightarrow O(n)$ work //

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(8) (Do first on white board)

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$$\hat{A} = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ x & x & x & x \end{pmatrix} \xrightarrow{G_{14}} \sim \begin{pmatrix} + & + & + & + \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & + & + & + \end{pmatrix} \xrightarrow{G_{24}}$$

$$\sim \begin{pmatrix} x & x & x & x \\ 0 & + & + & + \\ 0 & 0 & x & x \\ 0 & 0 & + & + \end{pmatrix} \xrightarrow{G_{34}} \sim \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{pmatrix} = \begin{pmatrix} R_2 & \gamma \\ 0 & \alpha \end{pmatrix} //$$

$x = R_2^{-1}\gamma$ is new solution

(9) $\|Ax-b\|_2^2 + \lambda^2\|x\|_2^2 = \left\| \begin{pmatrix} Ax-b \\ \lambda x \end{pmatrix} \right\|_2^2 =$

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$= \left\| \begin{pmatrix} A \\ \lambda I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2^2$

Reguler least squares problem. Parameter λ

small $\longleftarrow$ λ $\longrightarrow$ large

$\min \|Ax-b\|_2^2$
focus on accuracy

prevent a very noisy solution by minimizing $\|x\|_2^2$

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(10) A_m changes the image alot but

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$\kappa_2(A_m)$ is very small. A_b has a large condition number $\kappa_2(A_b) \approx 10^{20}$

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(11) $A = Q_1R_1 = Q_2R_2 \Rightarrow \underbrace{Q_2^TQ_1}_{\text{orthogonal}} = \underbrace{R_2R_2^{-1}}_{\text{triangular}}$

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$\Rightarrow R_2R_2^{-1} = D$ or $R_2 = DR_1$, $D = \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \ddots \\ & & & \pm 1 \end{pmatrix}$

Can change sign on rows of R