Lecture 4

(1) proof (x,λ) eigenpair $\Rightarrow Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$, $x \neq 0$ so non-trivial null space.

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(2) $$AX = A(x_1, .., x_k) = (Ax_1, .., Ax_k) = (\lambda_1 x_1, .., \lambda_k x_k) = (x_1, .., x_k)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}$$

$$\underset{A}{\boxed{n \times n}}\ \underset{X}{\boxed{n \times k}} = \underset{X}{\boxed{n \times k}}\ \underset{D}{\boxed{k \times k}}$$

If $X \in \mathbb{R}^{n \times n} \Rightarrow AX = XD \Rightarrow A = XDX^{-1}$

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(3) $(A^H)_{ij} = \overline{A_{ji}}$

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(4) proof If $A^H = -A$ and $Ax = \lambda x$ then

$$\lambda = \lambda x^H x = x^H(\lambda x) = x^H A x = x^H(-A^H x) = -(Ax)^H x$$
$$= -(\lambda x)^H x = -\bar{\lambda} x^H x = -\bar{\lambda}$$

So $\lambda = -\bar{\lambda}$ and $\mathrm{Re}\{\lambda\} = 0$

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(5) $p_J(\lambda) = \det(J - \lambda I) = (2-\lambda)^3$ so $\lambda = 2$

We have $\gamma_1(2) = 3$. Also $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the only eigenvector $\Rightarrow \gamma_2(2) = 1$.

Jordan decomposition $A = QDQ^T$, D = block diagonal.

(6) $p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -c_2-\lambda & -c_1 & -c_0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$

$= -\det\begin{pmatrix} -c_2-\lambda & -c_1 \\ 1 & -\lambda \end{pmatrix}(-\lambda) - \underbrace{\det\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}}_{1}(-c_0)$

$= c_0 + \lambda \underbrace{\det\begin{pmatrix} -c_2-\lambda & -c_1 \\ 1 & -\lambda \end{pmatrix}}_{\text{Rekursion step in general proof.}} = \dots = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$

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(7) proof $f(A) = \sum_{k=0}^{\infty} c_k A^k = \left\{ \begin{array}{l} A = XDX^{-1} \Rightarrow \\ A^k = XD\underbrace{X^{-1}X}_{I}D\underbrace{X^{-1}\cdots X}_{I}DX^{-1} \end{array} \right\}$

$= \sum_{k=0}^{\infty} c_k X D^k X^{-1} = X\left(\sum_0^{\infty} c_k D^k\right)X^{-1} =$

$= X \begin{pmatrix} \sum_0^{\infty} c_k \lambda_1^k & & \\ & \ddots & \\ & & \underbrace{\sum_0^{\infty} c_k \lambda_n^k}_{f(\lambda_n)} \end{pmatrix} X^{-1} = X f(D) X^{-1}$

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(8) proof Let (λ, x) be an eigenpair of A. Then

$Ax = \lambda x \Leftrightarrow (\lambda - a_{ii})x_i = \sum_{j\neq i} a_{ij}x_j, \quad i = 1, 2, \dots, n$

If $|x_k| = \max|x_i|$ then

$|\lambda - a_{kk}| \leq \sum_{j\neq i} |a_{ij}| \left|\frac{x_j}{x_k}\right| \leq \sum_{j\neq i} |a_{ij}| = r_i$

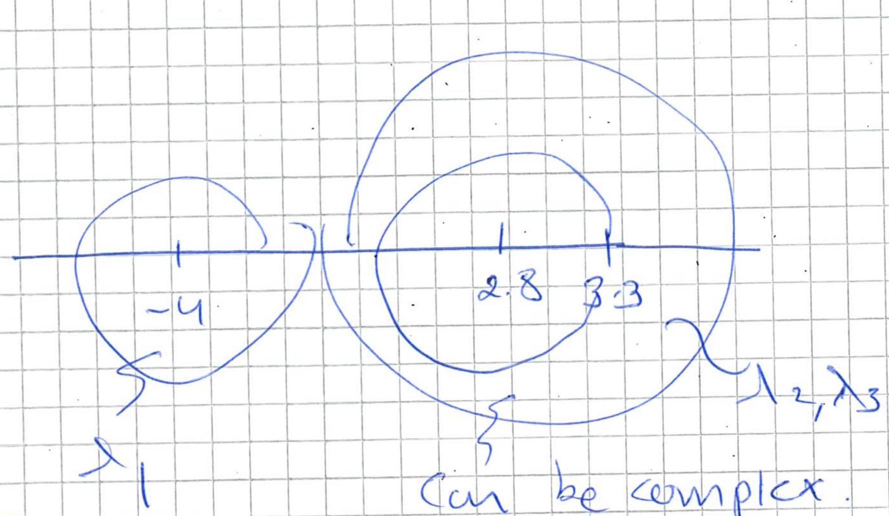
So λ belongs to one disc

(9) DBCS

$|3.3-\lambda| \leq 1.1$

$|2.8-\lambda| \leq 0.5$

$|-4-\lambda| \leq 0.5$



Looking at A^T gives same results

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(10) If $A\bar{x} \approx \lambda \bar{x} \Rightarrow \underbrace{\bar{x}}_{n\times 1} \cdot \underbrace{\lambda}_{1\times 1} = \underbrace{A\bar{x}}_{n\times 1}$

Normal equations: $\bar{x}^T \bar{x} \cdot \lambda = \bar{x}^T A \bar{x} \Rightarrow \lambda = \dfrac{\bar{x}^T A \bar{x}}{\bar{x}^T \bar{x}}$ //

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(11) Let $A = XDX^{-1}$, $D = \mathrm{diag}(\lambda_i)$, $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$

X is a basis for $\mathbb{R}^n$ so any vector can be written

$x = Xc = \sum_{i=1}^{n} c_i x_i$, $X = (x_1, x_2, \ldots, x_n)$

so

$$A^k x = \sum_{i=1}^{n} c_i A^k x_i = \sum_{i=1}^{n} c_i \lambda_i^k x_i = \lambda_1^k \left(c_1 x_1 + \sum_{2}^{\infty} c_i \left(\frac{\lambda_i}{\lambda_1}\right)^k x_i \right)$$

$$\Rightarrow \lambda_1^k c_1 x_1 \text{ //}$$

Power iteration

$x^{(k)} = A^k x$; $q^{(k)} = x^{(k)} / \|x^{(k)}\|_2$

$\Rightarrow q^{(k)} \to \pm x_1$ and $e_k = (q^{(k)})^T A q^{(k)} \to \lambda_1$

Error is dominated by $c_2 (\lambda_1/\lambda_2)^k x_2$ //

12 proof If $Ax = \lambda x$ then

$$(A - sI)x = (\lambda - s)x \Leftrightarrow \underbrace{\frac{1}{\lambda - s}}_{\mu} x = \underbrace{(A - sI)^{-1}}_{B} x$$

Apply power method to B instead.

Convergence rate

$$\frac{\mu_2}{\mu_1} = \frac{\frac{1}{(\lambda_i - s)}}{1/(\lambda_j - s)} = \frac{\lambda_j - s}{\lambda_i - s}$$

Very fast if $s \approx \lambda_i$ //