

Lecture 5

① proof If (x, λ) is an eigenpair of A then

$$Ax = \lambda x \Leftrightarrow \underbrace{XAX^{-1}}_B \underbrace{Xx}_y = \lambda \underbrace{Xx}_y \Leftrightarrow By = \lambda y$$

so (y, λ) eigen pair of B

② proof let $(\bar{\lambda}, \bar{x})$ be an approximate eigenpair.

Put $r = A\bar{x} - \bar{\lambda}\bar{x}$ and $B = A - \bar{\lambda}\bar{x}\bar{x}^T$

then

$$B\bar{x} = (A - \bar{\lambda}\bar{x}\bar{x}^T)\bar{x} = A\bar{x} - r \underbrace{\bar{x}^T\bar{x}}_{=1} = \bar{\lambda}\bar{x}$$

so $(\bar{x}, \bar{\lambda})$ exact eigenpair of B . Do a transform

$$X^{-1}BX = X^{-1}AX - X^{-1}r\bar{x}^T X = D \overset{\leftarrow \text{diag}(\lambda_i)}{+} F$$

Since $\bar{\lambda}$ is an eigenvalue of $D+F$ we get

$$(D+F)v = \bar{\lambda}v \Leftrightarrow v = (D - \bar{\lambda}I)^{-1}Fv$$

$$\Rightarrow \|v\|_2 \leq \frac{1}{\min |\lambda_i - \bar{\lambda}|} \|F\|_2 \cdot \|v\|_2$$

and

$$\min |\lambda_i - \bar{\lambda}| \leq \|F\|_2 = \|X^{-1}F\bar{x}\|_2 \leq \|X^{-1}\|_2 \cdot \|\bar{x}\|_2 \cdot \|F\|_2$$

③ proof If $(x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \lambda)$ is an eigenpair then

$$\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If $x_2 \neq 0$ then $A_2 x_2 = \lambda x_2$ so $\lambda \in \lambda(A_2)$. If instead $x_2 = 0$ then $A_1 x_1 = \lambda x_1$ so $\lambda \in \lambda(A_1)$ //

Also can start from $A_1 x_1 = \lambda x_1$ or $A_2 x_2 = \lambda x_2$ and construct eigenvectors for A (see problems).

④ proof (Induction) Suppose true for dim $n-1$.

For $A \in \mathbb{C}^{n \times n}$ we pick an eigen pair (λ, x) and find $X \in \mathbb{C}^{n \times (n-1)}$ so $U = \begin{pmatrix} x & X \end{pmatrix}$ is ~~orthogonal~~ unitary. Then

$$U^H A U = \begin{pmatrix} x^H & X^H \end{pmatrix} A \begin{pmatrix} x & X \end{pmatrix} = \begin{pmatrix} \lambda & \omega^H \\ 0 & C \end{pmatrix}$$

$\downarrow$
 $x^H A x = \lambda x^H x = 0$

Now use $C = U_1 T U_1^H$ to get

$$U^H A U = \begin{pmatrix} \lambda & \omega^H \\ 0 & U_1 T U_1^H \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & U_1 \end{pmatrix}}_Q \underbrace{\begin{pmatrix} \lambda & \omega^H \\ 0 & T \end{pmatrix}}_T \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & U_1 \end{pmatrix}}_Q^H$$

$$\Rightarrow A = Q T Q^H, \quad Q = U \tilde{U} //$$

(5) proof $A_{n+1} = R_n Q_n = Q_n^{-1} Q_n R_n Q_n = Q_n^{-1} A_n Q_n //$

(6) proof Use $A_{n+1} = Q_n^H A_n Q_n$

So $A_1 = Q_0^H A Q_0$; $A_2 = Q_1^H A_1 Q_1 = \underbrace{Q_1^H Q_0^H}_{S_1^H} A \underbrace{Q_0 Q_1}_{S_1}$; ...

and

$$A = Q_0 R_0$$

$$A^2 = Q_0 R_0 Q_0 R_0 = Q_0 A_1 R_0 = Q_0 Q_1 R_1 R_0 = S_1 \bar{R}_1$$

$$A^3 = S_2 \bar{R}_2 //$$

(7) 4×4 Hessenberg $\begin{pmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{pmatrix}$

(8) The QR step

Have $A_{n+1} = G_{34} G_{23} G_{12} A_n G_{12}^H G_{23}^H G_{34}^H$

Can apply rotations in any order.

$$G_{12} \begin{pmatrix} \boxed{x} & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{pmatrix} \sim \begin{pmatrix} + & + & + & + \\ 0 & \boxed{+} & + & + \\ 0 & x & x & x \\ 0 & 0 & x & x \end{pmatrix} \sim \begin{pmatrix} + & + & x & x \\ + & + & x & x \\ + & + & x & x \\ 0 & 0 & x & x \end{pmatrix}$$

Save $\boxed{+}$ used to create G_{23} before overwriting.

Continue with G_{23} and G_{34} the same way.

$$\textcircled{9} \quad Q_n^H A_n Q_n = \underbrace{Q_n^H (Q_n R_n + s_n I)}_I Q_n = R_n Q_n + s_n \underbrace{Q_n^H Q_n}_I$$

$$= A_{n+1} //$$

$$\textcircled{10} \quad \lambda_4 \approx 5.97 \text{ and } \lambda_3 \approx 3.81 \text{ gives}$$

$$\gamma = \left| \frac{5.97 - 6.05}{\underbrace{3.81 - 6.05}_{s_3}} \right| = 0.0357 \text{ or } \frac{1}{\gamma} \approx 28 //$$