F6

1 proof x^* fixed point means $x^* = x^* - \frac{f(x^*)}{f'(x^*)}$

$\Rightarrow f(x^*) = 0$ //

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2 proof $\|x^{(k+1)} - x^*\| = \|g(x^{(k)}) - g(x^*)\| \leq$

$|L| \, \|x^{(k)} - x^*\| \leq \dots \leq |L|^k \|x^{(0)} - x^*\|$

Since $|L| < 1$ we get $x^{(k)} \to x^*$ //

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3 proof Put $f(t) = g(x^* + t(x - x^*))$. Then

$$f'(t) = g'(x^* + t(x - x^*)) \cdot (x - x^*)$$

So

$$g(x) - g(x^*) = f(1) - f(0) = \int_0^1 f'(t)\,dt$$

$$= \int_0^1 g'(x^* + t(x - x^*)) \cdot (x - x^*)\,dt$$

and

$$\|g(x) - g(x^*)\| \leq \int_0^1 \|g'(x^* + t(x - x^*))\| \cdot \|x - x^*\|$$

$$\leq \max_{z \in \text{line}(x, x^*)} \|g'(z)\| \cdot \|x - x^*\|$$

Since $\|g'(x^*)\| < 1$ there is a $r > 0$ so

$\|g'(x)\| < 1$ for $x \in S_r$ //

Remark $\|g'(x)\| < 1$ in at least one norm is enough.

3

$$g(x) = x - f(x) = \begin{pmatrix} x_1 - x_1^2 - x_2^2 + 1 \\ x_2 - (x_1 - 0.5)^2 - x_2^2 + 1 \end{pmatrix}$$

$$\Rightarrow \quad g'(x) = \begin{pmatrix} 1 - 2x_1 & -2x_2 \\ -2(x_1 - 0{,}5) & 1 - 2x_2 \end{pmatrix}$$

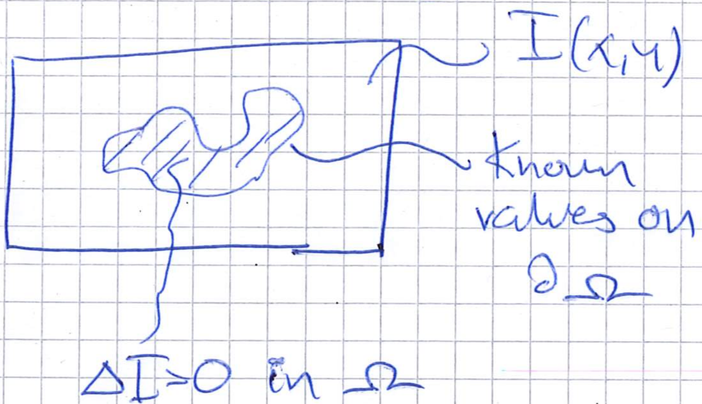
We have

$$x^* \approx \begin{pmatrix} 0.2486 \\ 0.9862 \end{pmatrix} \Rightarrow g'(x^*) \approx \begin{pmatrix} 0.5028 & -1.9369 \\ 0.5028 & -0.9369 \end{pmatrix}$$

$\Rightarrow \|g'(x^*)\|_2 \approx 2.2544$ but $\lambda_{1,2} = -0.2168 \pm 0.675i$

so $\rho(g'(x^*)) < 1$. There is a norm so $\|g'(x^*)\| < 1$.

4



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⑤ Example Let $f(x) = f(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$

→ Approximate f_1 by its linear part

$$f_1(x_1, x_2) = f_1\left(x_1^{(0)}, x_2^{(0)}\right) + \frac{\partial f_1}{\partial x_1}\cdot\left(x_1 - x_1^{(0)}\right) + \frac{\partial f_1}{\partial x_2}\left(x_2 - x_2^{(0)}\right)$$
$$+ O\left(|x_1 - x_1^{(0)}|^2 + |x_2 - x_2^{(0)}|^2\right)$$

So

$$f(x) = \begin{pmatrix} f_1(x_1^{(0)}, x_2^{(0)}) \\ f_2(x_1^{(0)}, x_2^{(0)}) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \underbrace{\begin{pmatrix} x_1 - x_1^{(0)} \\ x_2 - x_2^{(0)} \end{pmatrix}}_{x - x^{(0)}} + O\left(\|x - x^{(0)}\|^2\right)$$

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⑥ Problem Find x^* so that $f(x^*) = 0$. $f : \mathbb{R}^n \to \mathbb{R}^n$.

Suppose $x^{(k)} \approx x^*$. Approximate by Taylor series

$$f(x) \approx f(x^{(k)}) + J_f(x^{(k)})(x - x^{(k)}) = 0$$

$$\Rightarrow \quad J_f(x^{(k)})(x - x^{(k)}) = -f(x^{(k)})$$

$$\Rightarrow \quad x^{(k+1)} = x^{(k)} - J_f^{-1}(x^{(k)}) f(x^{(k)})$$

The error in each step is $O\left(\|x^{(k)} - x^*\|^2\right)$ so should get quadratic convergence

(9)

Tolerance $\|s^{(k)}\| \leq tol.$

Function values have error $\approx 10^{-9}$

$\Rightarrow$ Jacobian has errors $10^{-9}/10^{-4} \approx 10^{-5}$

$\Rightarrow$ Tolerance $\approx 10^{-4}$ might work.

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(10) proof Modify B_k so that $B_{k+1} = B_k + uv^T$

predicts the directional derivative in step k.

That is

$$B_{k+1} s^{(k)} = (B_k + uv^T)(x^{(k+1)} - x^{(k)}) = \underbrace{f(x^{(k+1)}) - f(x^{(k)})}_{y^{(k)}}$$

$$\Rightarrow \quad (v^T s^{(k)}) u = y^{(k)} - B_k s^{(k)}$$

We need $u \,/\!/\, y^{(k)} - B_k s^{(k)}$ so pick

$$u = \frac{y^{(k)} - B_k s^{(k)}}{\|s^{(k)}\|_2^2}$$

and find the smallest v so that

$$v^T s^{(k)} = \|s^{(k)}\|_2^2 \Rightarrow v = s^{(k)}$$