Lecture 8

1 proof Let x and y satisfy

$$Ax = \sigma y, \quad \sigma = \|A\|_2, \quad \|x\|_2 = \|y\|_2 = 1.$$

Find $V = (x, V_2)$ and $U = (y, U_2)$ orthogonal. Then

$$U^T A V = \begin{pmatrix} \sigma & \omega^T \\ 0 & B \end{pmatrix} = A_1$$

Also

$$\left\| A \begin{pmatrix} \sigma \\ \omega \end{pmatrix} \right\|_2^2 \geq \sigma^2 + \omega^T \omega$$

and since $\|A_1\|_2 = \|A\|_2 = \sigma$ we get $\omega = 0$ and

$$A = U \begin{pmatrix} \sigma & 0^T \\ 0 & B \end{pmatrix} V^T$$

Complete by induction.

2
$$\begin{pmatrix} \times & \times & 0 & 0 \\ 0 & \times & \times & 0 \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{pmatrix}$$
bidiagonal

3
$$G_{k,k+1} B^T B G_{k,k+1}^T = \underbrace{G_1 B^T G_1^T}_{\tilde{B}^T} \underbrace{(G_1 B G_1^T)}_{\tilde{B}}$$

4

Rank $(A) = k \Rightarrow A = \sum_{i=1}^{k} \sigma_i u_i v_i^T$

so

$$Ax = \sum_{i=1}^{k} (\sigma_i v_i^T x) u_i = \sum_{i=1}^{k} c_i u_i \in \text{span}(u_1, .., u_k) =$$

$$= \text{Range}(A)$$

If $Ax = 0$ then $v_i^T x = 0$ for $i = 1, .., k \Rightarrow$

$$\text{null}(A) = (\text{span}(v_1, .., v_k))^{\perp} = \text{span}(v_{k+1}, .., v_n)$$

Also

$$A^T = \sum_{i=1}^{n} \sigma_i v_i u_i^T$$

so

$$\text{Range}(A^T) = \text{span}(v_1, .., v_k) \text{ and } \text{null}(A^T) = \text{span}(u_{k+1}, .., u_m)$$

Have ON-basis for all relevant subspaces.

⑤ If $Ax=b$ and $A=U\Sigma V^T$ then

$$x = VV^Tx = \sum_{i=1}^{n} (v_i^Tx)v_i$$

and

$$Ax = \sum_{i=1}^{n} \sigma_i (v_i^Tx)u_i = \sum_{i=1}^{k} \sigma_i (v_i^Tx)u_i$$

Since $b \in \text{Range}(A)$ we have $b = \sum_{i=1}^{k} (u_i^Tb)u_i$

So $Ax=b$ if $(u_i^Tb) = \sigma_i (v_i^Tx) \Rightarrow$

$$x_1 = \sum_{i=1}^{k} \frac{u_i^Tb}{\sigma_i} v_i$$ is the component in $\text{null}(A)^{\perp}$.

General solution is $x = x_1 + x_2$, $x_2 \in \text{null}(A)$

so $x_2 = \sum_{k+1}^{n} c_i v_i$ //

How to check $b \in \text{Range}(A)$?

1) Compute $u_i^Tb = 0$ for ~~$i=$~~ $k+1, .., m$

or

2) Compute $b - \sum_{i=1}^{k} (b^Tu_i)u_i = 0$

Option 2 better if $m \gg k$ //

(6) Solve $\min \|Ax-b\|_2$, $A=U\Sigma V^T$ has rank k.

Divide

$$b = \underbrace{b_1}_{\in \mathrm{Range}(A)} + b_2 = \sum_{i=1}^{k}(b_i^T u_i)u_i + \underbrace{\sum_{k+1}^{m}(u_i^T b)u_i}_{\in \mathrm{Range}(A)^{\perp}}$$

We get

$$\|Ax-b\|_2^2 = \underbrace{\|Ax-b_1\|_2^2}_{\text{solve}} + \underbrace{\|b_2\|_2^2}_{\text{residual}}$$

The solution is

$$x = x_1 + x_2 = \underbrace{\sum_{i=1}^{k}\frac{u_i^T b}{\sigma_i}v_i}_{\text{unique}} + \underbrace{\sum_{k+1}^{n} c_i v_i}_{\text{null}(A)}$$

Thus

$$x_1 = A^{+}b = \left(\sum_{1}^{k}\frac{v_i u_i^T}{\sigma_i}\right)b$$

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(7)
$$V = (\underbrace{v_1,\dots,v_k}_{\text{null}(A)^{\perp}}, \underbrace{v_{k+1},\dots,v_n}_{\text{null}(A)}),$$

$$P = V_k V_k^T, \quad V_k = (v_1,\dots,v_k)$$

is the projection. Or

$$P = I - V_{n-k}V_{n-k}^T \quad V_{n-k} = (v_{k+1},\dots,v_n)$$

(8) $U = (U_k \; U_{m-k}) = (\underbrace{u_1, \dots, u_k}_{Range(A)}, u_{k+1}, \dots, u_m)$

The residual is

$$r = b - Ax = P_{Range(A)^\perp} b = U_{m-k} U_{m-k}^T b$$

or

$$Ax = P_{Range(A)} b = U_k U_k^T b$$

$$\Rightarrow \quad r = b - U_k U_k^T b$$

$U_k \in \mathbb{R}^{m \times k}$, $U_{m-k} \in \mathbb{R}^{m \times (m-k)}$

Pick the smallest matrix

(9) $Ax = b$ has no solution $\Rightarrow [A, b] \in \mathbb{R}^{m \times (n+1)}$ has rank $n+1$. Find the smallest $[E, r]$ so that

$$(A+E)x = b+r \Rightarrow (A+E, b+r)\begin{pmatrix} x \\ -1 \end{pmatrix} = 0$$

for some $x \in \mathbb{R}^n$. Compute

$$[A, b] = U \Sigma V^T, \quad \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{n+1} \end{pmatrix}$$

where $\sigma_{n+1} > 0$. Smallest perturbation is

$$[E, r] = -\sigma_{n+1} u_{n+1} v_{n+1}^T.$$

Now we have a null space and

$$\begin{pmatrix} x \\ -1 \end{pmatrix} \in \text{null}([A+E, b+r]) = \text{span}(v_{n+1})$$

so

$$x = -\frac{v_{n+1}(1:n)}{v_{n+1}(n+1)}$$